# NATIONAL OPEN UNIVERSITY OF NIGERIA

## SCHOOL OF SCIENCE AND TECHNOLOGY

**COURSE CODE: MTH 303**

**COURSE TITLE: Vector and Tensors**

# COURSE GUIDE

## Introduction

Vector and Tensors (MTH 303), requires the knowledge acquired in (MTH 142-Vector and Geometry) which you studied in our 100-Level., A good mastery of the course content in Mathematical Methods 1(MTH 281) will be helpful to learn this course successfully.

It is a three credit course. It is a compulsory course for any student majoring in mathematics at undergraduate level or B.Sc (Education) Mathematics. . It is also available to students offering Bachelor of Science (B.Sc) computer and Information & Communication Technology. Any student with sufficient background in mathematics can also offer the course if he/she so wish though it may not count as credit towards graduation if it is not a required course in his/her field of study.

The course is divided into two modules as enumerated below:

CONTENTS

## MODULE 1



MODULE 2

## What You Will Learn In This Course

This course guide tells you briefly what the course is about, what course materials you will be using and how you can work with these materials. In addition, it advocates some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully.

It gives you guidance in respect of our Tutor-Marked Assignment which will be made available in the assignment file. There will be regular tutorial classes that are related to the course. It is advisable for you to attend these tutorial sessions. The course will prepare you for the challenges you will meet in Vectors and Tensors

## Course Aim

The aim of the course is to provide you with an understanding of Vectors and Tensors. It also aims to make clear distinctions between the ways we handled problems in higher dimensional spaces , and provide you solutions to some problems that may arise in Engineering, physics, and other areas of human endeavour, where the knowledge of advance vector is required.

## Course Objectives

To achieve the aims set out, the course has a set of objectives. Each unit has specific objectives which are included at the beginning of the unit. You should read these objectives before you study the unit. You may wish to refer to them during your study to check on your progress. You should always look at the unit objectives after completion of each unit. By doing so, you would have followed the instructions in the unit.

Below are comprehensive objectives of the course as a whole. By meeting these objectives, you should have achieved the aims of the course as a whole. In addition to the aims above, this course sets to achieve some objectives. Thus, after going through the course, you should be able to:

- Prove continuity and establish limit functions in Vectors and Tensors
- Define Divergence, Gradient of Scalar Functions, and Curl of Vectors
- Establish Green's Theorem, Stokes's Theorems and Divergence Theorem
- Solve simple problems in Tensor Analysis.

## Working through this Course

To complete this course you are required to read each study unit, read the textbooks and read other materials which may be provided by the National Open University of Nigeria.

Each unit contains self-assessment exercises and at certain points in the course you would be required to submit assignments for assessment purposes. At the end of the course there is a final examination. The course should take you about a total of 16 weeks to complete. Below you will find listed all the components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course on time and successfully.

This course entails that you spend a lot of time to read and practice all related exercises. I would advice .that you avail yourself of the opportunities of the tutorial classes provided by the University

## Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your TMAs and attending tutorials. You should remember that you are required to submit all your assignments by the stipulated time and date. You should guard against falling behind in your work.

## Assessment

There are three aspects to the assessment of the course. The first is made up of self-assessment exercises, second consists of the tutor-marked assignments and third is the written examination/end of course examination.

You are advised to do the exercises. In tackling the assignments, you are expected to apply information, knowledge and technique you gathered during the course. The assignments must be submitted to your facilitator for formal assessment in accordance with the deadlines stated in the presentation schedule and the assignment file. The work you submit to your tutor for assessment will count for 30% of your total course work. At the end of the course you will need to sit for a final or end of course examination of about three hour duration. This examination will count for 70% of your total course mark.

## Tutor-Marked Assignment

The TMA is a continuous assessment component of your course. It accounts for 30% of the total score. You will be given four (4) TMAs to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs would be given to you by your facilitator and returned after you have done the assignment. Assignment questions for the units in this course are contained in the assignment file. You will be able to complete your assignment from the information and material contained in your reading, references and study units. However, it is desirable in all Degree level of education to demonstrate that you have read and researched more into your references, which will give you a wider view point and may provide you with a deeper understanding of the subject.

Make sure that each assignment reaches your facilitator on or before the deadline given in the presentation schedule and assignment file. If for any reason you can not complete your work on time, contact your facilitator before the assignment is due to discuss the possibility of an extension. Extension will not be granted after the due date.

## Final Examination and Grading

The end of course examination for MTH 303 (Vectors and Tensors) will be for about 3 hours and it has a value of 70% of the total course work. The examination will consist of questions, which will reflect the type of self-testing, practice exercise and tutor-marked assignment problems you have previously encountered. All areas of the course will be assessed.

Use the time between finishing the last unit and sitting for the examination, to revise the whole course. You might find it useful to review your self-test, TMAs and comments on them before the examination. The end of course examination covers information from all parts of the course.

## Course Marking Scheme

| Assignment | Marks |
|---|---|
| Assignments 1-4 | Four assignments, best three marks of the four count at 10% each -30% of course marks. |
| End of course examination | 70% of overall course marks. |
| Total | 100% of course materials. |

## Facilitators /Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on your progress and any difficulties you might face and provide assistance to you during the course. You are expected to mail your Tutor-Marked Assignment to your facilitator before the schedule date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary, hence you would have to contact your facilitator if:

- You do not understand any part of the study or the assigned readings
- You have difficulty with the self-tests
- You have a question or problem with an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating actively in discussions.

## Summary

MTH 303(Vectors and Tensors) is a course that intends to provide solutions to problems normally encountered by engineers, physicists and mathematicians in the course of doing their normal jobs. It also serves as a tool which often enables the mathematicians to widen the frontiers of their analytical, concerns to issues that have significant mathematical implications. Nevertheless, do not forget to apply the principles you have learnt to your understanding of Vectors and Tensors. I wish you success in the course and I hope that you will find it comprehensive and interesting.

# MODULE 1



## UNIT 1 ELEMENTARY VECTOR ALGEBRA

**CONTENTS**



## 1.0 INTRODUCTION

This course is an introductory work to Vectors and Tensors. In this unit 1, attempt will be made to revise some elements of vector algebra.

You should note that a vector quantity is distinguished from a scalar quantity by the fact that a scalar quantity possesses only magnitude, whereas a vector quantity possesses both magnitude and directions.

It is convenient to represent a vector geometrically as an arrow, pointing in the direction associated with the ones having length proportional to the associated magnitude. These and other properties of vector will be explained in this unit. You should read this unit carefully before proceeding to other unit.

## 2.0 OBJECTIVES

At the end of this unit, the students will learnt about:

- Vectors and scalar quantities
- Scalar product of two vectors
- Law of scalar product
- Vector products

Answer questions correctly on vector algebra.

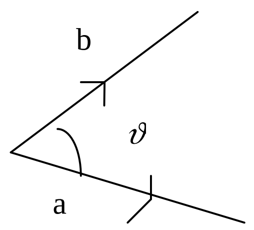
## 3.0 MAIN CONTENT

### 3.1 Revision of Elementary Vector Algebra

Definition: A scalar quantity has only magnitude while a vector quantity has both magnitude and direction.

Angle between two vectors.

Let $\underline{a}$ and $\underline{b}$ be two vectors such that $\underline{a}$ and $\underline{b}$ can be represented in the diagram below




Then the angle $\vartheta$ between the two vector in such that $0 \le \vartheta \le \pi$

#### 3.1.1 Scalar Product of Two Vectors

Let $\underline{a}$ and $\underline{b}$ be any two vectors (of the same dimension). The scalar production of $\underline{a}$ and $\underline{b}$ denoted by $\underline{a} \bullet \underline{b}$ is defined by $\underline{a} \bullet \underline{b} = |a| \; |b|$ Cos $\vartheta$ where $|a|$ and $|b|$ are the magnitude of vectors $\underline{a}$ and $\underline{b}$ respectively and $\vartheta$ is the angle between $\underline{a}$ and $\underline{b}$

$\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$ then $\underline{a} \bullet \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\therefore$ The angle between $\underline{a}$ and $\underline{b}$ is given by $\vartheta = Cos^{-1}\left(\frac{a_1b_1 + a_2\,b_2 + a_3\,b_3}{|\underline{a}|\;|\underline{b}|}\right)$

Provided $\underline{a}$ and $\underline{b}$ are non-zero vectors.

If $i, j, k$ are the usual unit vectors along the $x, y, z$ axis respectively then
$\underline{a} = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$
$\underline{b} = (b_b, b_b, b_3) = b_1 i + b_2 j + b_3 k$

Where $i \bullet j = j \bullet i = k \bullet i = i \bullet k = j \bullet k = k \bullet j = 0$ and $i \bullet i = j \bullet j = k \bullet k = 1$

### 3.1.2 Laws of Scalar Products

(i) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$
(ii) $m(\underline{a \bullet b}) = (m\underline{a} \bullet b) = (\underline{a} \bullet \underline{b})m = (\underline{a} \bullet \underline{b}m)$ $(m = any\ scalar))$
(iii) if $\underline{a} \bullet \underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendicular.

### 3.1.3 Vector Product

Let $\underline{a}$ & $\underline{b}$ be any two vectors with the same dimension and let $\hat{\underline{n}}$ be a unit vector perpendicular to both $\underline{a}$ & $\underline{b}$, the vector product of $\underline{a}$ & $\underline{b}$ written as $\underline{a} \wedge \underline{b}$ is defined by

$\underline{a} \wedge \underline{b} = |a|\ |b|$ Sin $\vartheta \hat{\underline{n}}$ , $0 \leq \vartheta \leq n$

Where $\vartheta$ is the angle between $\underline{a}$ & $\underline{b}$.

If $\underline{a} = a_1 i + a_2 j + a_3 k$
$\underline{b} = b_1 i + b_2 j + b_3 k$

Then

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2b_3 - a_3b_2) + j(a_3b_1 - a_1b_3) + k(a_1b_2 - a_2b_1)$$

### 3.1.4 Tripple Products

The scalar and vector products of 3 vectors $\underline{a}, \underline{b}$ and $\underline{c}$ may have meaningful products
$(\underline{a} \bullet \underline{b})\underline{c}$, $\underline{a} \bullet (\underline{b} \wedge \underline{c})$ and $\underline{a} \wedge (\underline{b} \bullet c)$

Hence the following laws are valid.

(i) $(\underline{a} \bullet \underline{b})\underline{c} \neq \underline{a}(\underline{b} \bullet \underline{c})$
(ii) $\underline{a} \bullet (\underline{b} \wedge \underline{c}) = \underline{b} \bullet (\underline{c} \wedge \underline{a}) = \underline{c} \bullet (\underline{a} \wedge \underline{b})$
(iii) $\underline{a} \wedge (\underline{b} \wedge \underline{c}) + (\underline{a} \wedge \underline{b}) \wedge \underline{c}$
(iv) $\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \bullet \underline{c})\underline{b} - (\underline{a} \bullet \underline{b})\ \underline{c}$
$(a \wedge b) \wedge c = (\underline{a} \bullet \underline{c})\underline{b} - (\underline{b} \bullet \underline{c})\ \underline{a}$
(v) $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$

**Remark 1**

1. The product $\underline{a} \bullet (\underline{b} \wedge \underline{c})$ is sometimes called the scalar triple product or box product and may be denoted by $[\underline{a}\ \underline{b}\ \underline{c}]$.

2. The product $a \wedge b \wedge c$ is called the vector triple product.

3. Reader are advised to provide proofs to i to v above by assuming that
$\underline{a} = a_1 i + a_2 j + a_3 k$
$\underline{b} = b_1 i + b_2 j + b_3 k$

**Remark 2**

(i) The quantity $\underline{a} \wedge \underline{b}$ is a vector quantity
(ii) If $\underline{a} \wedge \underline{b} = 0$ then at least one of $\underline{a}$ or $\underline{b}$ and then by implication the angle between them is zero.
(iii) If $\underline{a}$ and $\underline{b}$ are law then $\underline{a} \wedge \underline{b} = |a|\ |b|$
(iv) $\lambda(\underline{a} \wedge \underline{b}) = \lambda\underline{a} \wedge \underline{b} = \underline{a} \wedge (\lambda b)$ where $\lambda$ is any scalar.
(v) The magnitude of $\underline{a} \wedge \underline{b}$ and $|a \wedge b|$ is the area of the parallelogram with sides $\underline{a}$ and $\underline{b}$
(vi) If $\underline{a} \wedge \underline{b} = 0$ and neither $\underline{a} = 0$ nor $b = 0$ then $\underline{a}$ and $\underline{b}$ are parallel,

## 4.0 CONCLUSION

We have learnt about vector algebra in this unit. The materials in this unit are sufficient enough background, to go to the next unit.

## 5.0 SUMMARY

The following facts are to be remembered:

(i) That vector has magnitude and direction, unlike scalar quantity which has only magnitude
(ii) That two vectors can be multiplied in two ways.
(iii) Scalar products which result in scalar quantity
(iv) Vector product which result in vector quantity
(v) If $\underline{a} \wedge \underline{b} = 0$ and $\underline{a} \neq 0, \quad \underline{b} \neq 0$ then $\underline{a}$ is parallel to $\underline{b}$
(iv) That the magnitude of vectors whose product is $|\underline{a} \wedge \underline{b}|$ is the area of parallelogram with sides $\underline{a}$ and $\underline{b}$.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Find the length and direction comics of theorem $\underline{a}$ from the (1, -1, 3) to the midpoint of the line segment from origin to the point (6, -6, 4)
2. Prove that $|a+b| \leq |a| + |b|$ and $|a+b| \geq |a| - |b|$
3. if $\theta$ denotes the angle between the vectors $\underline{a}$ and $\underline{b}$ use a theorem in elementary geometry that $|a+b|^2 = |a|^2 + |b|^2 + 2|a|\,|b|$ Cos $\vartheta$

## 7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand: Advanced Calculus for Application 2nd Edition

# UNIT 2 VECTOR DIFFERENTIATION (GRADIENT, DIVERGENCE AND CURL)

**CONTENTS**



## 1.0 INTRODUCTION

In unit 1 we learnt about vector algebra, we established some properties of vectors.

In this unit, we shall consider vector differentiation and derive some important formula and properties of vector differentiation.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- differentiate vector quantities
- find the gradient of any scalar and vector field
- determine the curl and divergence of vector field
- solve correctly all exercises on the unit.

## 3.0 MAIN CONTENTS

**Definition 1**

The vector function $\underline{A}(u)$ is said to be continuous at a point $u_0$ if given $\varepsilon > 0$, we can find some $\delta(\varepsilon) > 0 \quad = \left|\underline{A}(u) - \underline{A}(u_0)\right| < \varepsilon$ whenever $\left|u - u_0\right| \leq \delta$

This is equivalent to $\frac{\lim}{u \to u_0} \underline{A}(u) = \underline{A}(u_0)$

**Definition 2**

The derivative of the vector function $\underline{A}(u)$ is given as $\frac{\lim it}{\Delta u \to 0} \frac{\underline{A}(u + \Delta u) - \underline{A}(u)}{\Delta u}$ provided that $\frac{d\underline{A}}{du} = \frac{dA_1 i}{du} + \frac{dA_2 j}{du} + \frac{dA_3}{du} k.$

Higher derivatives such as $\frac{d^2 \underline{A}}{d u^2}, \frac{d^3 \underline{A}}{d u^3}$ e.t.c. are similarly defined

**Remark 1**

If $\underline{A}, \underline{B}$ and $\underline{C}$ re differentiable vector functions of a scalar u and $\phi$ is a differentiable scalar functions then

(i) $\frac{d}{du}(\underline{A} + \underline{B}) = \frac{dA}{du} + \frac{dB}{du}$

(ii) $\frac{d}{du}(\underline{A} \bullet \underline{B}) = \underline{A}\frac{d\underline{B}}{du} + \underline{B}\frac{d\underline{A}}{du}$

(iii) $\frac{d}{du}(\underline{A} \wedge \underline{B}) = \underline{A} \wedge \frac{d\underline{B}}{du} + \frac{d\underline{A}}{du} \wedge \underline{B}$

(iv) $\frac{d}{du}(\phi \underline{A}) = \phi \frac{dA}{du} + \underline{A}\frac{d\phi}{du}$

(v) $\frac{d}{du}(\underline{A} \bullet (\underline{B} \wedge \underline{C})) = \underline{A} \bullet \frac{d}{du}(\underline{B} \wedge \underline{C}) + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C})$

$$= \underline{A}\left[\underline{B} \wedge \frac{d\underline{c}}{du} + \frac{d\underline{B}}{du} \wedge \underline{C}\right] + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C})$$

$$= \underline{A} \bullet \left(\underline{B} \wedge \frac{d\underline{c}}{du}\right) + A\left(\frac{d\underline{B}}{du} \wedge C\right) + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C})$$

(vi) $\frac{d}{du}(\underline{A} \bullet (\underline{B} \wedge \underline{C})) = \underline{A} \wedge \frac{d}{du}(\underline{B} \wedge \underline{C}) + \frac{d\underline{A}}{du} \wedge (\underline{B} \wedge \underline{C})$

$$= \underline{A} \wedge \left(\underline{B} \wedge \frac{dc}{du}\right) + \underline{A} \wedge \left(\frac{d\underline{B}}{du} \wedge c\right) + \frac{d\underline{A}}{du} \wedge (\underline{B} \wedge \underline{C})$$

## 3.1 Differentiation of Vectors

1. $\underline{A} = A_1 i + A_2 j + A_3 k$ then
   $d\underline{A} = dA_1 i + dA_2 j + dA_3 k$

(2) $d(\underline{A} \bullet \underline{B}) = \underline{A} \bullet d\underline{B} + d\underline{A} \bullet \underline{B}$

(3) $d(\underline{A} \wedge \underline{B}) = \underline{A} \wedge d\underline{B} + d\underline{A} \wedge B$

(4) if $\underline{A} = \underline{A}(n, y, z)$ then
$d\underline{A} = \frac{\partial \underline{A}}{\partial n} dn + \frac{\partial \underline{A}}{\partial y} dy + \frac{\partial \underline{A}}{\partial z} dz$

**Examples**

Given a vector $\underline{\phi}$ - Sin t i + Cos t j + t k

Obtain a. $\frac{d\underline{Q}}{dt}$ b. $\frac{d^2\underline{Q}}{dt^2}$ c. $\left|\frac{d\underline{Q}}{dt}\right|$ d. $\left|\frac{d^2\underline{Q}}{dt^2}\right|$

a. $\frac{dQ}{dt} = \frac{d}{dt}$ (Sin) i + $\frac{d}{dt}$ (Cost) j + $\frac{d}{dt}(t)k$

(b) $\frac{d^2Q}{dt^2} = \frac{d}{dt}\left(\frac{dQ}{dt}\right) = \frac{d}{dt}$ (Cos t i – Sin t j + k)

= - Sin t i – Cos t j

(c) $\left|\frac{dQ}{dt}\right| = \sqrt{Cos^2 t + Sin^2 t + 1^2} = \sqrt{Sin^2 t + Cos^2 t}$

= 0. 1

## 3.2 Gradient, Divergence and Curl of Vectors

Consider the vector operators $\nabla$ called "del" or nabla defined by

$$\nabla = i\frac{\partial}{\partial n} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

If $\phi(n, y, z$ has continuous first partial derivatives in a particular region, we define the gradient $\phi$ as:

$$\text{Grad } \phi = \nabla\phi = \left(i\frac{\partial}{\partial n} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$= i\frac{\partial\phi}{\partial n} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

**Examples**

(i) If $\phi(n, y, z) = nyz^2$ find $\nabla\phi$

$$\nabla\phi = i\frac{\partial\phi}{\partial n} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial n}\left(nyz^2\right) + j\frac{\partial}{\partial y}\left(nyz^2\right) + k\frac{\partial}{\partial z}\left(nyz^2\right)$$

$$= nz^2 i + nz^2 j + 2nyz\ k$$

(ii) if $\phi(n, y, z) = 3n^2 y - y^3 z^2$ find $\nabla\phi$

$$\nabla\phi = i\frac{\partial}{\partial n}\left(3n^2 y - y^3 z^2\right) + j\frac{\partial}{\partial y}\left(3n^2 y - y^3 z^2\right) + k\frac{\partial}{\partial z}\left(3n^2 y - y^3 z^2\right)$$

$$= 6nyi + \left(3n^2 - 3y^2 z^2\right)j - 2y^3 z\ k$$

(iii) If $\phi(n, y, z)$ is a scalar function P, we know that

$$\text{Grad } \phi = \nabla\phi = i\frac{\partial\phi}{\partial n} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

And $d\underline{r} = dni + dyj + dzk$

$$\text{Grad } \phi \bullet d\underline{r} = \frac{\partial \phi}{\partial n}\, dn + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z}\, dz$$
$$= d\phi$$

This implies

$$d\phi = \text{ grad } \phi \bullet dr = \nabla \phi \bullet dr.$$

### 3.2.1 Directional Derivatives

We recall $d\phi = d\underline{r} \bullet \nabla \phi \quad , |d\underline{r}| = ds$

$\Rightarrow \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{|d\underline{r}|}$ hence $\frac{d\underline{r}}{ds}$ is the unit vector in the direction of $d\underline{r}$

$\therefore \frac{d\phi}{ds} = \frac{d\underline{r}}{ds} \bullet$ grad $\phi$ , if $\frac{d\underline{r}}{ds} = \hat{a}$

Then $\frac{d\phi}{ds} = \hat{a} \bullet$ grad $\phi$

**NOTE:**

$\frac{d\phi}{ds}$ is the propagation of grad $\phi$ on the unit vector $\hat{a}$ and the directional derivative of $\phi$ in the direction of $\hat{a}$

**Remark**

(i) The unit Normal vector N = $\frac{\nabla \phi}{|\nabla \phi|}$

(ii) The grad of sum and products of scalars.

$$\nabla(\underline{A + B}) = i \frac{\partial}{\partial n}(\underline{A} + \underline{B}) + j \frac{\partial}{\partial y}(\underline{A} + \underline{B}) + k \frac{\partial}{\partial z}(\underline{A} + \underline{B})$$
$$= \left( \frac{\partial \underline{A}}{\partial n} i + \frac{\partial \underline{A}}{\partial y} j + \frac{\partial \underline{A}}{\partial z} k \right) + \left( \frac{\partial \underline{B}}{\partial n} i + \frac{\partial \underline{B}}{\partial y} j + \frac{\partial \underline{B}}{\partial z} k \right)$$
$$= \nabla \underline{A} + \nabla \underline{B}$$

$$\nabla(\underline{A} \bullet \underline{B}) = i \frac{\partial}{\partial n}(\underline{A} \bullet \underline{B}) + j \frac{\partial}{\partial y}(\underline{A} \bullet \underline{B}) + k \frac{\partial}{\partial z}(\underline{A} \bullet \underline{B})$$

$$= i\left(\frac{\partial \underline{A}}{\partial n} \bullet \underline{B} + \underline{A} \bullet \frac{\partial \underline{B}}{\partial n}\right) + j\left(\frac{\partial \underline{A}}{\partial y} \bullet \underline{B} + \underline{A}\frac{\partial \underline{B}}{\partial y}\right) + k\left(\frac{\partial \underline{A}}{\partial z} \bullet \underline{B} + \underline{A}\frac{\partial \underline{B}}{\partial z}\right)$$

$$= \left(i\frac{\partial \underline{A}}{\partial n} \bullet \underline{B} + j\frac{\partial \underline{A}}{\partial y} \bullet \underline{B} + k\frac{\partial \underline{A}}{\partial z} \bullet \underline{B}\right) + \left(i\underline{A} \bullet \frac{\partial \underline{B}}{\partial n} + j\underline{A} \bullet \frac{\partial \underline{B}}{\partial y} + k\underline{A} \bullet \frac{\partial \underline{B}}{\partial z}\right)$$

$$= \left(i\frac{\partial \underline{A}}{\partial n} + j\frac{\partial \underline{A}}{\partial y} + k\frac{\partial \underline{A}}{\partial z}\right) \bullet \underline{B} + \underline{A} \bullet \left(i\frac{\partial \underline{B}}{\partial n} + j\frac{\partial \underline{B}}{\partial y} + k\frac{\partial \underline{B}}{\partial z}\right)$$

$$= \underline{B}(\nabla \underline{A}) + \underline{A}(\nabla \underline{B})$$

## 3.2.2 Divergence of a Vector

The divergence of a vector function $\underline{A}(n, y, z)$ is defined by dir $\underline{A} = \nabla \bullet A$

$$= \left(i\frac{\partial}{\partial n} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \bullet (A_1 i + A_2 j + A_3 k)$$

$$= \frac{\partial A_1}{\partial n} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

**Examples**

1. Obtain the divergence of the vector function $\underline{A} = 2nzi + yzj - ny^2k$ at the point (1, -1, 1)

2. If $\phi = n^2yz$ a scalar function and $\underline{A} = 2nz\,i + yzj - ny^2k$ Find $\nabla \bullet (\phi A)$ at the point (1, -1, 1)

3. Determine the content a so that the vector $\underline{v} = (n + 3y)i + (y - 2n)j + (n + az)k$ is solenoid

**Solution**

1. $$\nabla \bullet \underline{A} = \left(\frac{\partial}{\partial n}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \bullet (2nzi + yzj - ny^2k)$$

$$= \frac{\partial}{\partial n}(2nzi) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(ny^2)$$

= 2z + z = 3z

2. $$\nabla \bullet (\phi A) = \nabla(2n^3yz^2i + n^2y^2z\ \ j - n^3y^3z\,k)$$

$$= 6n^2yz^2 + 2n^2yz^2 - n^3y^3$$

A vector $\underline{v}$ is solenoid of its divergence zero
$\therefore \nabla \bullet \underline{V} = 0$ implies that
1 + 1 + a = 0 $\Rightarrow$ a = - 2

### 3.2.3 The Curl of a Vector Function

The curl of a vector function $\underline{A}$ define denoted by curl $\underline{A} = \nabla \wedge A$

$$= \left( i\frac{\partial}{\partial n} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \wedge \left( A_1 i + A_2 j + A_3 k \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial n}, \frac{\partial A_2}{\partial n} - \frac{\partial A_1}{\partial y} \right)$$

**Examples**

1. Find the curl of $\underline{A}$ if $\underline{A} = 3n^2 y\, i + y\, z^2 j - n\, z\, k$
2. If $\phi = n^2 y^2$ and A = $2\, n\, z\, i + y\, z\, j - n\, y^2 k$
   Obtain curl $(\phi\, A)$

**Solution**

1. Curl A = $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3n^2 y & yz^2 & -nz \end{vmatrix} = \left(0 - 2yz,\ +z,\ 0 - 3n^2\right)$

$$= 2yzi + zj - 3n^2 k$$

2. $\phi\ A = 2n^3 yz^2 i\ +\ n^2 y^2 z^2 j - n^3 y^2 z\ k$

$$\nabla \wedge A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2n^3 yz^2 & n^2 y^2 z^2 - & n^3 y^2 z \end{vmatrix}$$

$$= \left(2ny^2 z^2 - 2n^3 z^2,\ 4n^3 yz + 3n^2 y^2 z, 2ny^2 z - 2n^3 z^2\right)$$

This section is very important for the understanding of the remaining units. You are to master this unit very well.

## 5.0 SUMMARY

We have established formula to the following:

(i) Vector differentiation
(ii) The grad $\phi$ of scalar field $\phi$
(iii) The curl of a vector function
(iv) We show that the unit normal vector N = $\frac{\nabla\phi}{|\nabla\phi|}$

## 6.0 TUTOR-MARKED ASSIGNMENTS

1. A particle moves along the curve $n = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5$ where t is the time. Find the component of its velocity and acceleration at time t = 1

2. If $\underline{A} = 5t^2 i + tj - t^3 k$
   And $\underline{B} =$ Sin t i – Cos t j
   Find (a) $\frac{d}{dt}\left(\underline{A} \bullet \underline{B}\right)$
   (b) $\frac{d}{dt}\left(\underline{A} \wedge \underline{B}\right)$
   (c) $\frac{d}{dt}\left(\underline{A} \bullet \underline{A}\right)$

3. If $\phi(n, y, z) = ny^2 z$ and $\underline{A} = nzi - ny^2 j + yz^2 k$
   Find $\frac{\partial^3(\phi A)}{\partial^2 n \partial z}$ at point (2, -1, 1)

4. Let $r = (x, y, z)$be a vector.
   Prove that $\nabla \bullet \left(\nabla \bullet |r|^m\right) = 3m\, r^{m-2}$

5. Find the divergence of the vector
   $\underline{B} = \left(y^2 - 2xyz^3, 3 + 2xy - x^2 z^3, 6z^3 - 3x^2 yz^2\right)$

6. If $\underline{A} = \left(2x^2 + 8xy^2 z, 3x^3 y - 3ny, 2x^3 z\right)$, show that $\underline{A}$ is not solenoidal.

7. Show that the vector $\underline{A} = \frac{r}{r^3}$ is irotational where r = xi + yj + zk

8. If $\underline{A} = x^2 yi + y^2 zj - z^2 xk$
show that $\nabla \wedge (\nabla \wedge A) = \nabla(\nabla \bullet A) - \nabla^2 A.$

9. Given that $\underline{E}$ and $\underline{H}$ are two vectors which are assumed to have continuous partial derivatives with respect to position and time. Furthermore,

Suppose $\nabla \bullet \underline{E} = 0, \quad \nabla \bullet \underline{H} = 0$

$$\nabla \wedge \underline{E} = \frac{-1}{c}\frac{\partial \underline{H}}{\partial t} \; ; \; \nabla \wedge \underline{H} = \frac{1}{c}\;\frac{\partial \underline{E}}{\partial t}$$

Prove that $\underline{E}$ & $H$ satisfy the equation

$$\nabla^2 \phi = \frac{1}{c^2}\;\frac{\partial^2 \phi}{dt^2}$$

Use the above relations to show that

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\left(E^2 H^2\right)\right] + c\nabla\left(\underline{E} \wedge \underline{H}\right) = 0$$

## 7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand: Advanced Calculus for Application 2nd Edition

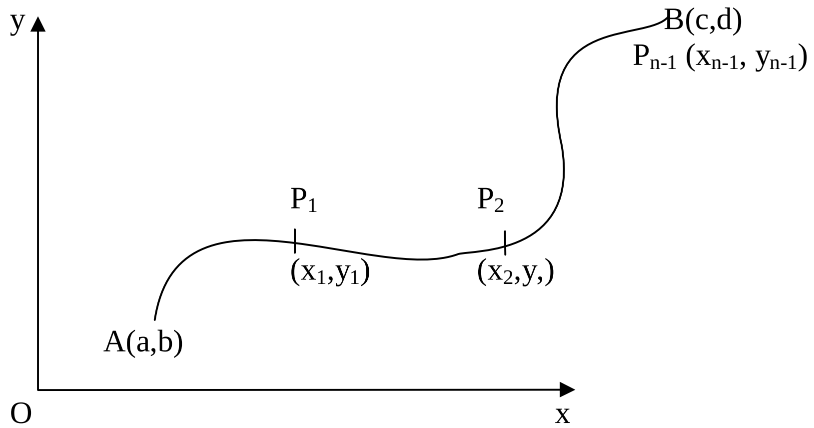
# UNIT 3 THE LINE INTEGRAL

## CONTENTS



## 1.0 INTRODUCTION

In a line integral unlike other integrals, we have to consider two or more functions at a time, for integration purposes. Suppose these two functions are $M(x, y)$ and $N(x, y)$ such that they are single valued and continuous at every point of a curve AB. Divide the curve AB into $\pi$ parts by means of $Pi(x_i, y_i)$ $i = 1,2,3...........\pi - 1$




Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$ where $x_0 = b, x_n = c, y_n = d$

Let $(\xi, n)$ be defined by

$x_{i-1} \leq \varsigma \prec x_{i-1}$ $y_{i-1} \leq \eta \leq yi$

We form the product and then add then to get

$$\sum_{i-1}^{n} \{M(\zeta, \pi)\nabla x_i + N(\zeta, \eta)\nabla y_i\}$$

Limit of this sum $\alpha\pi \to \infty$ and all $\nabla x_i \to 0, \nabla y \to 0$ simultaneously is defined as $\alpha$ line integral along the curve A$\beta$ of two functions M and N simultaneously.

Thus, we write

$$\lim_{\substack{\pi \to \infty \\ \Delta x_i \to 0 \\ \Delta y_i \to 0}} \left[\sum_{i-1}^{n} \{M(\zeta,\pi)\Delta x_i + N(\zeta,\eta)\Delta y_i\}\right] =$$

$$\int_{CurveAB} [M(x,y)dx + N(x,y)dy]$$

## 2.0 OBJECTIVES

At the end of this unit, you should be able to :

. To evaluate the line integral of vector functions

- Solve problems relating to line integrals

## 3.0 MAIN CONTENT

### 3.1 The Line Integral

Let $\underline{A}(x,y,z) = A_1 i + A_2 j + A_3 k$ be a vector function of position defined and continuous along a curve C. The integral of the tangential component of $\underline{A}$ along C is written as:

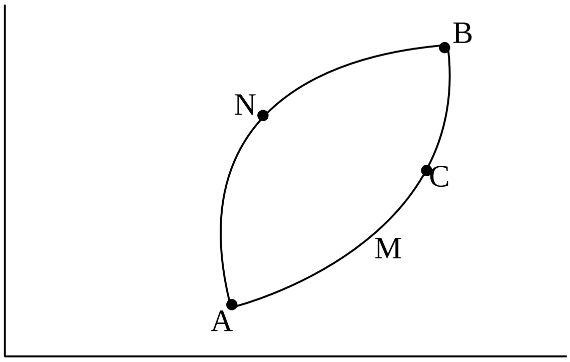
$$\int_c \underline{A} \bullet d\underline{r} = \int_c A_1 dn + A_2 dy + A_3 dz$$

Where $d\underline{r} = dx_i + dyj + d\underline{z}k.$

For instance in Aerodynamic and fluid mechanics this line integral $\int_c \underline{A} \bullet d\underline{r}$ is called the circulation of $\underline{A}$ about c

Where $\underline{A}$ represents the velocity of Air or the velocity of the fluid as the case may be.

Line Integral about closes plane




Evaluate the integral I = $\oint_c (2xydy + x^2dx)$ where C made up of the 3 sides of triangle motor vertices $0(0,0)$, $A(1,0)$, & $B(1,1)$ Ans = $1/3$.

**SELF ASSESSMENT EXERCISE**

1. Evaluate the line integral $I = \int_c ydx \ from \ -a \ to \ a$ where C is the circle $x^2 + y^2 = a$

2. Find the work done in moving a particle once around a circle C in the xy-plane if the circle C has centre at the origin and radius If the force field $\underline{F} = (2x - y + z)i + (x + y - z)i + (3x - 2y + 4z)k$ is as given.

3. If $\underline{F} = (x - 3y)i + (y - 2x)j$ and C is the closed curve in the xy-plane, n=2 cost y = 3 sint, t=0 to t= $2\pi$. Evaluate $\int \underline{F}dr$.

4. Find the total work done if a particle is moved in a force field by $\underline{F} = 3xyi - y^2 j$ along the curve $y = 2x^2$ in the $xy$-plane from (0, 0) to (1, 2).

5. If $\underline{A} = (5x^2 + 4y)i - 14yzj + 20xz^2k$. Evaluate $\int \underline{A} \bullet d\underline{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths.

i. $c = t,\ y = t^2,\ z = t^3$

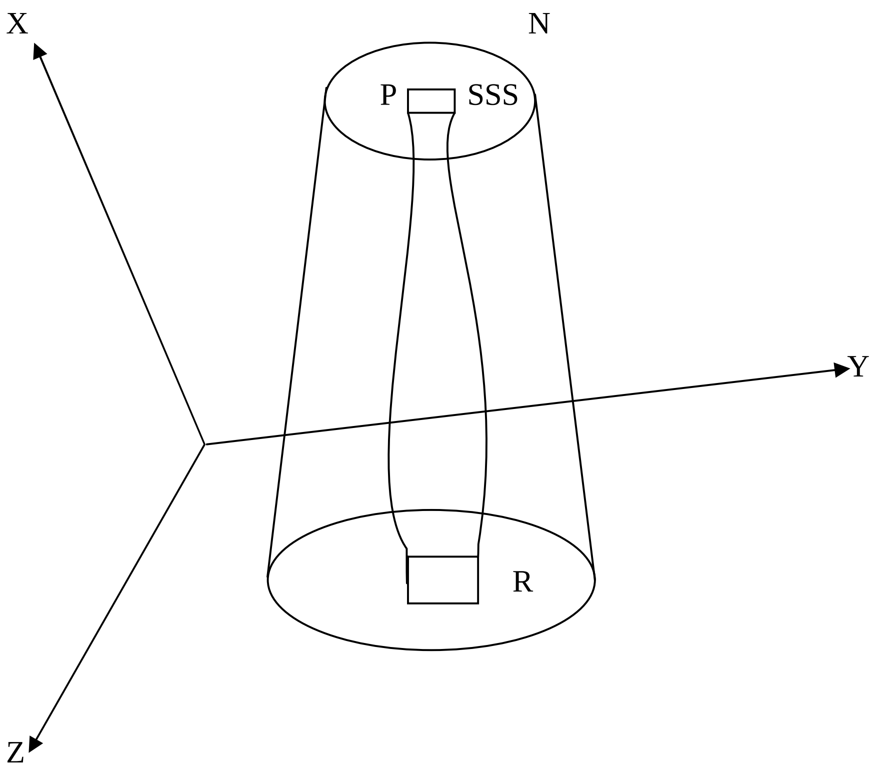
ii. The straight line from (0, 0, 0) to (1, 0, 0) then from (1, 0, 0) to (1, 1, 0) and then to (1, 1, 1)

6. Calculate $\int_c \underline{v} \bullet d\underline{r}$ where V= $2ji + 3xj$ and c is given by

i. The straight line joining (2, 0) to (0, 0)

ii. The arc of a circle with centre t the origin and radius 2 units

### 3.1.1 The Surface Integral




Let $\delta A$ represent an element of R and $\delta S$ the corresponding of area of S at the point $P(x, y, z)$ on S. Let also $\phi(x, y, z)$ be a function of position on S and let Y denote the angle between two outward normal PN to the surface P and the positive Z axis.

$\delta A\ \alpha\ \delta S\ Cos\ Y \Rightarrow \delta S\ \alpha\ \delta A Sec\ Y$

And

$\Sigma\phi(x, y, z)\delta S$ is the total value of $\phi(x, y, z)$ taken over the surface $\varsigma\infty\delta_n \rightarrow 0$. This sum becomes the integral

$$I = \int_{\varsigma} \phi(x, y, z) ds = \int\int_{R} (x, y, z) \ Sec \ y \, dxdy$$

Sec y = $\sqrt{1 + z_n^2 + zy^2}$

This implies that

$$I_s = \int_{s} \phi ds = \int\int_{R} \phi \sqrt{1 + z_n + z_y^2 \ dxdy}$$

Let S be a 2 sided surface and let one side be considered arbitrary as the positive side associated with the differential of the surface area *ds*. A vector $d\underline{s}$ whose magnitude is $ds$ and direction is that of $\hat{\underline{n}}$. then $ds = \hat{\underline{n}} ds$ where $\hat{\underline{n}}$ is the limit vector normal to any point of the positive side of S. the integral

$\int_{C} \underline{A} \bullet d\underline{s} = \int_{S} \underline{A} \bullet \hat{\underline{n}} ds$ is an example of a surface called the flux of $\underline{A}$ over S. others are

i. $\int\int \phi ds$

ii. $\int\int \phi \hat{\underline{n}} ds$

iii. $\int\int \underline{A} \wedge d\underline{s}$

we have $d\underline{A} = ds \quad Sec \ Y.$

where if $z = f(n, y)$ implying that $z - f(n, y) = 0$

$$\hat{\underline{n}} = \frac{\nabla[z - (-cn, y)]}{|\nabla[z - f(n, y)]|} = \frac{-\frac{\partial f}{\partial n} i - \frac{\partial f}{\partial y} j + k}{\sqrt{+_{\alpha}^2 + f_y^2 + 1}}$$

This implies that

$$\nabla[z - f(n, y)] = \left(\sqrt{f_n^1 + f_y^2 + 1}\right) \hat{\underline{n}} = ds$$

If the angle Y between the z-axis and $\underline{\hat{n}}$ is acute the positive sign is adopt and if the angle is obtuse we adopt the negative sign.

Hence

$$\iint \underline{A} \bullet \underline{\hat{n}}\, ds = \iint \underline{\underline{A}} \bullet \underline{\hat{n}} \frac{d_n dy}{\left|\underline{\hat{n}} = k\right|}$$ depending on the projection

The projection

**Example 1**

$\iint \underline{A} \bullet \underline{\hat{n}}\, ds$ where $\underline{A} = 18i - 12j + 3yk$ and S is that part of the plane, $2n + 3y + 6z = 12$ which is in the $ny$ plane ( $\ln -$ quadrate)

$$\iint \underline{A} \bullet \hat{n}\, ds = \iint A.\hat{n} \frac{dndy}{\left|\hat{n} \bullet k\right|}$$

$$\underline{\hat{n}} = \nabla(2n + 3y + 6z - 12) = 2i + 3j + 6k$$

$$\underline{\hat{n}} = \frac{2i + 3j + 6k}{7} = \left(\tfrac{2}{7}, \tfrac{3}{7}, \tfrac{6}{7}\right)$$

$$\underline{\hat{n}} \bullet k = \left(\tfrac{2}{7}, \tfrac{3}{7}, \tfrac{6}{7}\right) = (0,0,1) = \tfrac{6}{7}$$

$$\underline{A} \bullet \underline{\hat{n}}(18z, -12, 3y) \bullet \left(\tfrac{2}{7}, \tfrac{3}{7}, \tfrac{6}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

Now $2n + 3y + 6z = 12$ gives $z = \dfrac{12 - 2n - 3y}{6}$

$$\therefore \underline{A} \bullet \underline{\hat{n}} = \frac{6(12 - 2n - 3y) - 36 + 18y}{7}$$

$$= \frac{72 - 12n - 18y - 36 + 18y}{7} = \frac{36 - 12n}{7}$$

This implies that

$$\iint A \bullet \hat{n} ds = \iint \underline{A} \bullet \underline{\hat{n}} \frac{dndy}{\left|\underline{\hat{n}} \bullet k\right|} = \iint \frac{36 - 12n \bullet}{7} \frac{7ndy}{6}$$

$= \int\int (6-2n)dndy$

To obtain the limit of integration of n we set y = z = 0, and for y we set z = 0

Hence we have

$$\int_{1}^{6} \int_{y=0}^{\frac{12-2n}{3}} (6-2n)dndy = 24$$

**SELF ASSESSMENT EXERCISE**

1. Evaluate $\iint \underline{A} \bullet \hat{\underline{n}}\, ds$ where $A = zi + nj - 3y^2 z\ k$ and S is the surface of the cylinder $n^2 + y^2 = 16$ included in the 1st quadrant between z = 0 and z = 5

2. Evaluate $\iint \phi \hat{\underline{n}} ds$ where $\phi = 3ny^2 z$ and C is the surface $n^2 + y^2 = 16$ included in the 1st quadrant between z = 0 and z = 5.

3. If $F = yi + (n - 2nz)j - nyk$. Evaluate $\iint (\nabla \wedge F).\hat{n}\, ds$ where S is the surface of the sphere $n^2 + y^2 + z^2 = a^2$ from the $ny -$ plane.

### 3.1.2 The Volume Integral

The volume or space integral is given by

$\iiint \underline{A} \bullet dr \quad or \quad \iiint \phi\, dr \quad for \quad a$

Closed surface

**Example**

If $F = n^2 i + zj + yzk$. Evaluate $\int\int_s F.ds = \iiint_S \nabla F \bullet dr$ where V is the volume enclosed by the cube given by $0 \le n \le 1,\ 0 \le y \le 1\ \ and\ \ 0 \le z \le 1$.

**Solution**

$$\iint F \bullet ds = \iiint \nabla s \bullet \nabla \bullet F\, dr$$

Where $dr = dndydz$ and $\nabla \bullet F = 2n + y$

$$\iiint \nabla \bullet F \;\; dr = \int_0^1 \int_0^1 \int_0^1 (2n + y) dndydz = \tfrac{2}{3}$$

## 4.0 CONCLUSION

The materials in this unit are very important for the understanding of subsequent units. You must understand this work thoroughly before moving to the next section.

## 5.0 SUMMARY

In this unit, we established the relationship between line integral of scalar field and vector functions.

Thus, we have use the line integral in finding

(i) Surface integral
(ii) Volume integral or space integral which is given as:
$\iiint \underline{A} \bullet dr \quad or \quad \iiint \phi\, dr, \quad for \;\; a$ closed surface.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Let $F = 2nxi - nj + y^2 k$ . Evaluate $\iiint \underline{F} \bullet dv$ where V is the region bounded by the surface $n = 0,\; y = 0,\; z = n^2, z = 4,\; x = 2,\; y = 6$

2. If $F = ny^2 i + yzj + zn^2 k$. Evaluate $\iint F \bullet ds$ over the sphere given by $n^2 + y^2 + z^2 = 1$

3. Evaluate $\iint (\nabla \wedge A) \bullet ds$ where $A = (n - z)i + (-nz - y)j + (y^2 + 2z)k \;\; and \;\; S$ is the surface of the sphere having centre at (3, -1, 2) and radius 3.

4. If $F = \left(2n^3 - 3z, -4ny, -4n\right)$. Evaluate $\iiint \nabla \bullet F \, d\underline{r}$ where $\underline{r}$ is the closed region bounded by the plane $n = 0, y = 0, z = 0$ and $2n + 2y + z = 4$

## 7.0 REFERENCES/FURTHER READINGS

F.B. Hildebrand: Advanced Calculus for Applications.

P.D.S. Verma: Engineering Mathematics

## MODULE 2



# UNIT 1 GREEN'S THEOREM, DIVERGENCE THEOREM AND STOKES' THEOREM

**CONTENTS**



## 1.0 INTRODUCTION

In this unit, we will discuss some mathematical theorems such as Green's theorem, divergence theorem and stokes theorem.

These theorems are useful in handling equations of mathematical physics, particularly in the area of velocity of a fluid in three dimensions.

We know on application of these theorems particularly the divergence theorem that the velocity of an incompressible fluid has zero divergence.

Some other applications will be considered in this section.

The references at the back of this unit throw more light on the applications of these theorems; you may wish to contact them.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to apply:

- the Green's theorem
- divergence theorem and stokes theorem to solving problems arising from mathematical physics.

## 3.0 MAIN CONTENT

### 3.1 Green's Theorem

If R is a closed region on the $ny$-plane bounded by a simple closed curve C and if M and N are single valued functions which are continuous in n and y having continuous derivatives in R, then

$\oint Mdn + Ndy = \iint_R \left( \frac{\partial N}{\partial n} - \frac{\partial M}{\partial y} \right) dndy$ where the curve C is traversed counter clockwise

**Example**

Verify the Green's theorem in the plane for $\oint_c \left( ny + y^2 \right) dn + n^2 dy$ where C is a close curve of the region bounded by $y = n$ and $y = n^2$

**Solution**

The point of intersection of $y = n$ and $y = n^2$ is (0, 0) and (1, 1), $M = ny + y^2$ and $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ and $\frac{\partial N}{\partial x} = 2n$

$\Rightarrow \oint \left( ny + y^2 \right) dn + n^2 dy = \iint_R (2n - x - 2y) dndy$

L.H$^{-}S$ is $\oint_c \left( ny + y^2 \right) d\, n + n^2 dy$ along $y = n^2, dy = 2ndx$

$\therefore I_1 = \int \left( n^3 + n^4 + 2n^3 \right) dn = \int_0^1 \left( 3n^3 + n^4 \right) = \frac{19}{20}$

Along $y = n,\ dy = dn$

$I_2 = \int_0^1 \left( n^2 + n^2 + n^2 \right) dn \quad = -3\int_0^1 n^2 dn = -1$

$I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$

Now R.H.S.

$$\iint_R (n-2y)dndy = \int_0^1 \int_{n^2}^{n} (n-2y)dydn$$

$$= \int_0^1 \left(ny - y^2\right)\int_{n^2}^{2} dn = \int_0^1 \left(n^4 - n^3\right)dn = -1/20$$

**SELF ASSESSMENT EXERCISE**

Evaluate having the Green's theorem the integral $(y - Sin\ n)dn + Cos\, ndy$ where C is bounded by (0, 0), $(\bar{1}/_{2,}0)$, $(\bar{\pi}/_2, 1)$

## 3.2 Divergence Theorem

If V is the volume bounded by a closed surface A and $\underline{A}$ is a vector function of position with continuous derivatives then

$$\iiint_v \nabla A dr = \iint_s \underline{A} \bullet \hat{n}\, ds \text{ or } \int_v div\ Fdv = \iiint_v \nabla F dv = \int_s \underline{F} \bullet d\underline{s} = \iint_R F \bullet \hat{}_n ds$$

**Example**

Verify the divergence theorem for
$\underline{A} = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by $n^2 + y^2 = 4,\ z = 0$ and $z = 3$

$$\text{L.H.}^{-}S = \iiint_v \nabla \bullet dv = \iiint \frac{\partial}{\partial n}(4n) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= \iiint (4 - 4y + 2z)dr$$

$$\int_{-2}^{2} \int_{-\sqrt{4-n^2}}^{\sqrt{4-n^2}} \int_0^3 (4 - 4y + 2z)dzdydx = \delta 4\prod$$

R.H.S. The surface S of the cylinder consists of a base $S_1(z=0)$ and the top $S_2(z=3)$ and the convex position $S_3\left(n^2 + y^2 = 4\right)$. Then the surface is given by

$$\iint_s \underline{A} \bullet \hat{\underline{n}}\, ds = \iint_{s_1} \underline{A} \bullet \hat{\underline{n}} ds_1 + \iint_{s_2} \underline{A} \bullet \hat{\underline{n}}\, ds_2 + \iint_{s_3} \underline{A} \bullet \hat{\underline{n}}\, ds_3$$

On $S_1(z=0)$ $\underline{n} = -k$, A = $4xi - 2y^2 j$ $\underline{A} \bullet \hat{\underline{n}} = 0$

On $S_2(z=3)$ n = 4, $A = 4x_i - 2y^2 j + 9k$ so that

$$\iint_{S_2} A \bullet \hat{n} ds = \iint 9 ds_2 = 9\iint ds_3 = 9(4\Pi) = 36\Pi$$

On $S_3(x^2 + y^2 = 4)$

$$\nabla(x^2 + y^2 = 4) = 2xi + 2yj \Rightarrow \hat{\underline{n}} = \frac{xi + yj}{2}$$

$$\underline{A} \bullet \hat{\underline{n}} = (4xi - 2y^2 j + z^2 k) \bullet \left(\frac{xi + yj}{2}\right)$$

$$= 2x^2 - y^3$$

Let $x = 2\ Cos\,\vartheta$, $y = 2\ Sin\ \vartheta$ $ds_3 = 2d\vartheta dz$

This implies that

$$\iint_{S_3} \underline{A} \bullet \hat{\underline{n}} ds_3 = \int_0^{2\Pi} \int_0^3 \left(2(2\ Cos\,\vartheta) - (2\ Sin\,\vartheta)^3\right).\ 2\ dzd\vartheta$$

$$= \int_0^3 \int_0^{2\Pi} \left[8\ Cos^2\vartheta - 8\ Sin^3\vartheta\ \right] = dzd\vartheta$$

$$= 16 \int_0^{2\Pi} \int_0^3 (Cos^2\vartheta - Sin^2\vartheta) dzd\vartheta$$

$$= 48\int_0^{2\Pi} (Cos^2\vartheta - Sin^3\vartheta) d\vartheta = 48\Pi$$

Hence the total surface $S = 0 + 36\Pi + 48\Pi = 84\Pi$

**SELF ASSESSMENT EXERCISE**

1. Evaluate $\iint_S \underline{F} \bullet \underline{n}\, ds$ where $\underline{F} = 4xzi - y^2 j + yzk$ and S is the surface of the cube bounded by $n = 0$, $x = 1$, $y = 1$ $z = 0$, $z = 1$

2. Evaluate $\iint \underline{r} \bullet \underline{n}\, ds$ where S is a closed surface.

## 3.3 Stokes' Theorem

If S is an open two-sided surface bounded by a closed non-intersecting curve C (Simple closed curve) then if A has continuous derivatives.

$$\oint_c \underline{A} \bullet d\underline{r} = \iint_s (\nabla \wedge A) \bullet \hat{\underline{n}} ds = \iint_c (\nabla \wedge A) \bullet d\underline{s}$$

Where C is traversed counter-clock wisely

**Example:**

Use stokes' theorem to determine

$\iint_s (\nabla \wedge A) \bullet \hat{\underline{n}}\, ds$ where $A = (y - z + 2)i + (yz + 4)j - xzk$
And S is the surface of the closed cube
$n = 0, y = 0,\ z = 0,\ \ x = 2,\ y = 2,\ z = 2$ the $xy-$plane.

**Solution:**

By stokes' theorem

$$I = \iint_s (\nabla \wedge \underline{A}) \bullet \hat{\underline{n}}\, ds = \oint_c \underline{A} \bullet d\underline{r}$$

$$= \oint_c (y - z + 2)dx + (yz + 4)dy - xzdz$$

Since it is above the $xy-$plane then $z = 0$

$$\therefore I = \oint_c (y + 2)dx + 4dy$$

By applying the Green's theorem

$$M = y + 2,\ N = 4\ \frac{\partial M}{\partial y} = 1 \frac{\partial N}{\partial x} = 0$$

$$\therefore I = \iint - dydx = -\int_0^2 \int_0^2 dxdy = -4.$$

**SELF ASSESSMENT EXERCISE**

Use the stokes' theorem to determine
$\oint_s (2x^2 - y)dx + y^{2z3}dy + y^{3_z^2}\, dz$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ is the $xy-$plane.

## 4.0 CONCLUSION

The materials in this unit should be well understood before proceeding to tensor Analysis which made use of all the materials we have developed in the proceeding chapters.

## 5.0 SUMMARY

We have established the following formula

1. $\oint Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$

2. $\iiint \nabla A dr = \iint_S \underline{A} \bullet \underline{\hat{n}}\, ds.$

3. $\oint A \bullet dr = \iint (\nabla \wedge A) \bullet \underline{\hat{n}}\, ds = \iint_C (\nabla \wedge A) \bullet d\underline{s}$ .

Study the material presented in the preceding sections because the whole ideas will be used in the subsequent sections on vector and tensors.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate by using Green's theorem

   $\int_C \left[ (x^2 + y)dx + (x - y^2)dy \right]$

   Where C is a closed curve formed by $y^3 = x^2$ and $y = x$ between (0, 0) and (1, 1)
2. Use stokes' theorem to transform $\int ydx + xdz + zdy$ to a surface integral.
3. Prove that

   Grad $(\phi\psi) = \phi\, grad\, \psi + \psi\, grd\, \phi$
4. Prove that

   div $(\phi\mu) = \phi\, divu + u.grad\, \phi$

## 7.0 REFERENCES/FURTHER READINGS

P.D.S. Verma: Engineering Mathematics
F.B. Hildebrand: Advanced Calculus for Application

# UNIT 4 TENSOR ANALYSIS

**CONTENTS**



## 1.0 INTRODUCTION

This section introduces the students into Tensor Analysis. The work will make use of th ideals developed in the previous units.

## 2.0 OBJECTIVES

At the end of this unit you should be able to:

- define tensors
- Perform various operations on Tensors
- Solve correctly any exercises on Tensor analysis.

## 3.0 MAIN CONTENT

A point in the N-dimensional space is a set of N numbers denoted by $\left(X^1, X^2, ......., X^N\right)$ where 1, 2,….,N are taken not s exponents but as superscripts. The fact that we cannot visualize points in spaces of dimension higher that 3 has of course nothing to do with the existence of such space.

## 3.1 Transformation of Co-ordinates

Let $X^1, X^2, X^3, ..., X^n$ and $\overline{X}^1, \overline{X_2}, \overline{X}_3, ...\overline{X}^N$ be co-ordinates of point in two different frames of references. Suppose there exist independent relations between the co-ordiante of the two systems having the form

$$\begin{aligned} \overline{X}^1 &= \overline{X}^1\left(X^1, X^2, ..., X^N\right) \\ \overline{X}^2 &= \overline{X}^2\left(X^1, X^2, ..., X^N\right) \\ &-------------- \\ \overline{N}^N &= \overline{X}^N\left(X^1, X^2, ...X^N\right) \end{aligned} \quad \text{----------- (1)}$$

Which can be indicated briefly as $\overline{X}^R = \overline{X}^R\left(X^1, X^2, ...X^N\right)$ _________ $(\overline{n})$ where K = 1, 2, …N. It is assumed that the functions involved are singled valued, continuous and have continuous derivatives. The converse to each set of coordinates $\left(\overline{X}^1, \overline{X}^2, ..., \overline{X}^N\right)$ the will correspond a unique set $\left(X^1, X^2, ..., X^N\right)$ given by $X^K - X^k\left(\overline{X} \ \ , \overline{X}^2, \overline{X}^N\right)$ ______-_ (iv)

The relation given by equation (ii) and (iii) defines a transformation of co-ordinate from one frame of reference to another frame of reference.

## 3.2 Cartesian Tensor

Definition: When the transformations are from one rectangular co-ordinate system to another the tensors are called Cartesian tensors.

## 3.3 Summation Convection

Note that $\sum_{i=1}^{N} aj\, x^j = a_1 x^1 + a_2 x^2 + ... + a_N x^N$ but a shorter notation is simply to write as $a_j x^j$ where the convention that whenever the index (super or subscript) is repeated in a given term we are to sum from the index from 1 to N unless otherwise specified. This is called the summation convention instead of moving the index j we could us another letter say P and the sum could be written as $a_p X^p$. Any index which is repeated in a given term so that the summation convection applied is called a dummy index.

As index occurring only once is called “free index” and can stand for any of the number 1, 2, …,N such as k is equation (ii) and (iii)

**Examples**

1. If $\phi = \phi\left(X^1, X^2, ..., X^3\right)$ then the differential of $\phi$

$$\partial\phi = \frac{\partial\phi dx^1}{\partial x_1} + \frac{\partial\phi}{\partial X^2}dx^2 + ... + \frac{\partial\phi}{\partial X^N}dx^N$$

$$\therefore d\phi = \sum_{j=1}^{N}\frac{\partial\phi}{\partial X^j}dx^j = \frac{\partial\phi}{\partial X^j}dx^j$$

2. $$\frac{d\overline{X}^k}{\partial t} = \frac{\partial\overline{X}^k}{\partial X^1}\frac{\partial X^1}{\partial t} + \frac{\partial\overline{X}^k}{\partial X^2}\frac{\partial X^2}{\partial t} + .... + \frac{\partial\overline{X}^k}{\partial X^N}\frac{\partial N^N}{\partial t}$$

$$\frac{\partial\overline{X}^k}{\partial t} = \frac{\partial\overline{X}^k}{\partial X^M} \bullet \frac{\partial X^M}{\partial t}$$

3. If $\partial S^2 = g_{11}\left(\partial X^1\right)^2 + g_{22}\left(\partial X^2\right)^2 + g_{33}\left(\partial X^3\right)2$

$$\partial S^2 = g_{kk}\left(\partial X^k\right)^2$$

## 3.4 Contraviant and Covariant Vectors

If N quantities $A^1, A^2, ...A^N$ is in a co-ordinate system $\left(X^1, X^2, ..., X^N\right)$ are related to N other quantities $\overline{A}^1, \overline{A}^2, ..., \overline{A}^N$. In another co-ordinate system $\overline{X}^1, \overline{X}^2, ..., \overline{X}^N$. By the transformation equations

$$\overline{A}^P = \sum_{p=1}^{N}\frac{\partial\overline{X}^P}{\partial X^2}A\text{q}$$

They are called component of a Contraviant vector or Contraviant tensor of the first rand or first order.

If N quantities $A_1, A_2, ..., A_N$ in a co-ordinate system $X^1, X^2, ..., X^N$ are related to N, other quantities $\overline{A}_1, \overline{A}_2, ..., \overline{A}_N$ in another co-ordinate system $\overline{X}^1, \overline{X}^2, ..., \overline{X}^N$. By the transformation equalities

$$\overline{A}_P = \sum_{q=1}^{N}\frac{\partial x^2}{\partial\overline{x}^p}A_q, P = 1,2,..., N$$

By convection $\overline{A}_P = \dfrac{\partial x^q}{\partial\overline{x}^p}Aq$ they are called component of a covariant vector or tensor of the first rank or first order.

Note: That a superscript is used to indicate contravariant component an subscript is used to indicate a covariant component.

Examples on Contravariant and Covariant Tensor

1. Write the law of transformation for:

(i) $\overline{A}_{jk} = \frac{\partial \overline{x}^i}{\partial x^a} \bullet \frac{\partial x^b}{\partial \overline{x}^j} \bullet \frac{\partial x^c}{\partial \overline{x}^k} = A^k_{bc} \Rightarrow$

(a) Illustrate: As an aid for remembering the transformation, note that the relative position of indices $p_1 q_2 r$ On the left side of the transformation are the same as those on the right hand side of the equation. Some these indexes are associated with $\overline{x}$ co-ordinate and since indices $i, j, k$ are easily written as it was done

(b) $B^{mn}_{ijk}$ Consider $\overline{B}^{pq}_{rst} = \frac{\partial \overline{x}^p}{\partial x^m}, \frac{\partial \overline{x}^q}{\partial x^n}, \frac{\partial x^i}{\partial \overline{x}^r}, \frac{\partial x^j}{\partial \overline{x}^s}, \frac{\partial x^k}{\partial \overline{x}^t} B^{mn}_{ijk}$

(c) $C^m$ Consider $\overline{C}^p \frac{\partial \overline{x}^p}{\partial x^m} C^m$

## 3.5 Contravariant, Covariant and Mixed Tensor

If $N^2$ quantities $A^{qs}$ in a co-ordinate system $(X^1, X^2, ..., X^N)$ are related to $N^2$ other quantities $\overline{A}^{pr}$ in another co-ordinate system $(\overline{X}^1, \overline{X}^2, ..., \overline{X}^N)$ by transformation equations $\overline{A}^{pr} = \sum_{q=1}^{N} \sum_{j=1}^{N} \frac{\partial \overline{x}^p}{\partial x^2} \bullet \frac{\partial \overline{x}^r}{\partial x^s} A^{qs} \quad p, r = 1,2,..,N$

$\overline{A}^{pr} = \frac{\partial \overline{x}^p}{\partial x^q} \bullet \frac{\partial \overline{x}^r}{\partial x^s} A^{qs}$ it is called contravariant component of a tensor of the second rank. The $N^2$ quantities $A_{qs}$ are called covariant component of a tensor of the second rank is $\overline{A}_{pr} = \frac{\partial x^q}{\partial \overline{x}^p} \bullet \frac{\partial x^s}{\partial \overline{x}^r} A_{qs}$ e.g. of mixed tensor $\delta^p_q$

Scalar or invariant: A scalar or invariant is called a tensor of rank zero.

**Symmetric and Skew Symmetric Tensor**

A tensor is called symmetric w.r.t. of indices if its component remains unaltered upon. Interchange of the indices e.g. if $A^{mpt}_{qs} = A^{pmr}_{qs}$ the tensor is symmetric in p and m.

A tensor is called skew symmetric with .respect .to. 2 contravariant or 2 covariant indices of its component change sing upon interchange of the indices e.g. $A^{mpr}_{qa} = -A^{pmr}_{qs} \left(in\ m \& p\right)$

## Tensor of rank greater than 2

$A^{qst}_{kl}$ are component of a mixed tensor of rank 5. (Contravariant of order 5 and covariant of order 2).

### The kronecker Delta

This written $\delta^j_k$ is defined as $\delta^j_k = \begin{Bmatrix} 0 & If\ j = k \\ 1 & If\ \ j = k \end{Bmatrix}$ is a mixed tensor of 2nd rank.

### Examples

1. Evaluate (a) $\int^p_q A^{qr}_s$ (b) $\int^p_q \int^q_r$

### Solution

Since $\int^p_q = 1\ \ if \quad p = q\ \ \&\ \ 0\ \ if \quad p = q$

$\Rightarrow \int^p_q A^{qr}_s = A^{qr}_s$

2. Show that $\dfrac{\partial x^p}{\partial x^q} = \int^p_q$

### Solution

If p = q, then $\dfrac{\partial x^p}{\partial x^q}$ =1 since $X^p = X^q$

If p = q then $x^p$ and $x^q$ re independent then $\dfrac{\partial x^p}{\partial x^2} = 0 \Rightarrow \dfrac{\partial x^p}{\partial x^q} = \int^p_q$

## 3.6 Fundamental Operation with Tensor

1. Addition and Subtraction: The sum of 2 or more tensor of the same rank and type i.e. same number of Contraviant indices and same number of covariant indices is also a tensor of the same rank and type thus if $A_q^{mp}$ and $B_q^{mp}$ are the tensor then $C_q^{mp} = A_q^{mp} + B_q^{mp}$ of vector is Enumerative and Associative. The difference of 2 tensor of the same rank and type is also a tensor of the same rank and type. i.e. $D_q^{mp} = A_q^{mp} - B_q^{mp}$

**Example**

If $A_r^{pq}$ and $B_r^{pq}$ are tensor. Prove that their sum and difference are tensor.

**Solution**

$$\overline{A}_i^{jk} = \frac{\partial \overline{x}^j}{\partial x^p}, \frac{\partial \overline{x}^k}{\partial x^q}, \frac{\partial x^r}{\partial \overline{x}^j} A_r^{pq}$$

$$\overline{B}_i^{jk} = \frac{\partial \overline{x}^j}{\partial x^p} \cdot \frac{\partial \overline{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \overline{x}^j} B_r^{pq}$$

$$\Rightarrow \overline{A}_i^{jk} + \overline{B}_i^{jk} = \left( \frac{\partial \overline{x}^j}{\partial x^p} \cdot \frac{\partial \overline{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \overline{x}^i} \right) A_r^{pq} + B_r^{pq}$$

And $\overline{A}_i^{jk} - \overline{B}_i^{jk} = \left( \frac{\partial \overline{x}^j}{\partial x^p} \cdot \frac{\partial \overline{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \overline{x}^j} \right) A_r^{pq} - B_r^{pq}$

## 3.7 Outer Multiplication

The product of 2 tensor is a tensor whose rank of the given tensor. This product which involve ordinary multiplication of the component of the tensor is called the outer product

e.g. $A_q^{pr} \bullet B_s^m = C_{qs}^{prm}$ is the outer product of $A_q^{pr}$ and $B_s^m$

Contraction: If one contravariant and one covariant index of a tensor are set equal, the result indicates that, summation over the equal. Indices are to be taking according to the summation convection. This resulting sum is a tensor of rank 2 lesser than the original tensor. The process is called contraction.

**SELF ASSESSMENT EXERCISE**

1. If a tensor of rank 5 $A_{qs}^{mpr}$ i.e. set r = s to obtain $A_{qs}^{mpr} = B_{q}^{mp}$

**Tensor of Rank 3**

Exercise 2 By setting the above $B_{q}^{mp}$ where p = 2 $B_{q}^{mp} = D^{m}$

**Internal Multiplication**

By the process of outer multiplication of 2 tensor followed by contraction, we obtain a new tensor called an inner product of a given tensor, the process is called inner multiplication.

e.g. if $A_{q}^{mp}$ and $B_{st}^{r}$ the outer product $A_{q}^{mp} \quad B_{st}^{r}$, letting q = r, we obtain $A_{r}^{mp}$ and $B_{st}^{r}$ and this is the inner product, moreover, if p = s then another inner product is obtain $A_{q}^{mp}$ and $B_{pt}^{r}$

**Tensor form of gradient, divergent and curl**

(i) Gradient: If $\phi$ is a scalar of invariant, the gradient of $\phi$ is defined by $\nabla\phi$ = grad$\phi = \delta$, $P = \dfrac{\partial\phi}{\partial x^{p}}$ where $\phi$, P is the covariant derivative of $\phi \quad w.r.t \quad x^{p}$

(ii) Divergent: The divergent of $A^{p}$ is the contraction of its covariant derivative $w.r.t \quad x_{p}^{q}$ be the contraction of $A_{q}$

Mathematically,

$$\text{Dir } A^{p} = A_{q}^{p} = \frac{1\partial}{\sqrt{g\partial x^{k}}}\left(\sqrt{g}A^{k}\right) = \frac{\partial A^{p}}{\partial X^{q}}$$

(iii) Curl: The curl of $A_{p}$ is $A_{p,q} - A_{q,p} = \dfrac{\partial A_{p}}{\partial X^{q}} - \dfrac{\partial A_{q}}{\partial X^{p}}$

Tensor of rank 2, the curl is also define as $\sum^{pqr} \quad A_{p,q}$

(iv) Laplacian: The Laplacian of $\phi$ is the divergence of grad $\phi$ and $\nabla^{2}\phi$ = dir $\phi, P = \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial Xj}\left(\sqrt{g} \;\; g^{jk}\dfrac{\partial\phi}{\partial x^{k}}\right)$.

In case $g \prec 0, \sqrt{g}$ must be replace by $\sqrt{-g}$ both cases $g \succ 0$ & $g \prec 0$ can be included the written $\sqrt{|g|}$ in place $\sqrt{g}$ .

Alternating Tensor $\sum_{i,j,k}$

This is define by $\sum_{i,j,k} = \begin{cases} i & if \quad i,j,k \ are \ in \ cycle \\ -1 & if \quad i,j,k \ are \ anti \ cycle \\ 0 & if \quad i,j,k \ on \quad 0 = k \end{cases}$

$i - j = \delta ij$ $\qquad 0 - i = \delta_{i-j} = 1$
$i - k = \delta ik$ $\qquad i.j = \delta_{i-j} = 0$ for $i + j$
$j - k = \delta_{jk}$
$i \wedge j = |i|.|j|$ sum O.K where $\phi$ is the angle between $i$ & $j$
$\delta_r \quad \phi = \bar{\pi}/_2$
$|i| = 1, \quad |j| = 1$
$i \wedge j = 1 - 1$ Sum $\frac{\pi}{2} K = K$

Using alternating tensor

$$i \wedge j - \sum_{K_{21}}^{3} \sum_{ijk} = \left[\sum_{ij1} + \sum_{ij2} + \sum_{ij3}\right] k$$
= (0 + 0 + 1)K = K

**Notation**

Let us denote the unit tensor by $\delta$, $\delta$ is a matrix.
$\underline{\delta}_1$ = (1, 0, 0) = I a unit vector along k – arms
$\underline{\delta}_2$ = (0, 1, 0) = j a unit vector along y – arms
$\underline{\delta}_3$ = (0, 1, 0) = k a unit vector along z – arms

$$\underline{\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{unit 3 x 3 matrix}$$

Use alternating tensor to establish the result of $\underline{u} \wedge \underline{v}$ where $\underline{u} \wedge \underline{v}$ are vectors. By vector product

$$\underline{u} \wedge \underline{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\ 0 - (u_1 v_3 - u_3 v_1) j \ + \ (u_1 v_2 - u_2 v_1) k$$

By alternating tensor

$$\underline{u} \wedge \underline{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \Sigma_{ijk}\, \underline{\delta i}\ U_j V_k$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ \Sigma_{ij1}\, \underline{\delta}_i U_j V_1 + \Sigma_{ij2}\, \underline{\delta}_i U_j V_2 + \Sigma_{ij3}\, \underline{\delta}_i U_j V_3 \right]$$

$$= \sum_{j=1}^{3} \Sigma_{i21}\, \underline{\delta}_i U_2 V_1 + \Sigma_{i31}\, \underline{\delta}_i U_3 V_1 + \Sigma_{i12}\, \underline{\delta}_i U_1 V_2 + \Sigma_{i32}\, \underline{\delta}_i U_3 V_2 + \Sigma_{i13}\, \underline{\delta}_i U_1 V_3 + \Sigma_{i21}\, \underline{\delta}_i U_2 V_3$$

$$= \left[ \sum_{i=1}^{3} \Sigma_{i21}\, \underline{\delta}_i U_2 V_1 + \Sigma_{i31}\, \underline{\delta}_i U_3 V_1 + \Sigma_{i12}\, \underline{\delta}_i U_1 V_2 + \Sigma_{i32}\, \underline{\delta}_i U_3 V_2 + \Sigma_{i13}\, \underline{\delta}_i U_1 V_3 + \Sigma_{i23}\, \underline{\delta}_i U_2 V_3 \right]$$

$$\Sigma_{321}\, \underline{\delta}_3 U_2 V_1 + \Sigma_{231}\, \underline{\delta}_2 U_3 V_1 + \Sigma_{312}\, \underline{\delta}_3 U_1 V_2 + \Sigma_{132}\, \underline{\delta}_1 U_3 V_2 + \Sigma_{213}\, \underline{\delta}_2 U_1 V_3 + \Sigma_{123}\, \underline{\delta}_1 U_2 V_3$$

$$= \Sigma_{321}\, \underline{\delta}_3 U_2 V_1 + \Sigma_{231}\, \underline{\delta}_2 U_3 V_1 + \Sigma_{312}\, \underline{\delta}_3 U_1 V_2 + \Sigma_{132}\, \underline{\delta}_1 U_3 V_2 + \Sigma_{213} \underline{\delta}_2 U_1 V_3 + \Sigma_{123}\, \underline{\delta}_1 U_2 V_3$$

$$= -U_2 V_1 K + U_3 V_1 j + U_1 V_2 K - U_3 V_2 i - U_1 V_3 j + U_2 V_3 i$$

$$= (U_2 V_3 - U_3 V_2) i + (U_3 V_1 - U_1 V_3) j + (U_2 V_1) k = \underline{U} \wedge \underline{V}$$ proved.

Defined

The vector $\underline{U} = (U_1, U_2, U_3)$, $\delta_1 = (1,0,0)$, $\delta_2 = (0,1,0), \delta_3 = (0,0,1)$ e.t.c. are called first order tensor.

Let $L^-$ be a second order tensor

$$L^- = \begin{pmatrix} L^-_{11} & L^-_{12} & L^-_{13} \\ L^-_{21} & L^-_{22} & L^-_{23} \\ L^-_{31} & L^-_{32} & L^-_{33} \end{pmatrix} \ \& \ \delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_{3x3}$$

## 3.8 Multiplication of Tensors

We have various multiplications

Dynoadic product

Let $\underline{U} = (U_1, U_2, U_3)$, $\underline{V} = (V_1, V_2 V_3)$ be vectors i.e. first order tensor

Definition: we defined the dynoadic products of $\underline{u}$ & $\underline{v}$ written as $\underline{U}\ \underline{V}$

$$\underline{U}\ \underline{V} = \begin{pmatrix} U_1V_1 & U_1V_2 & U_1V_3 \\ U_2V_1 & U_2V_2 & U_2V_3 \\ U_3V_1 & U_3V_2 & U_3U_3 \end{pmatrix}$$

Hence $\delta_1 - \delta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ & $\delta_1 - \delta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Let $L^{-1} = \delta_i \delta_j L^{-}{}_{ij}$ $\quad N = 3$

Let s be scalar and $L^{-}$ be second order tensor then

$$SL^{-} = S\delta_i \delta_j (i) \quad N = 3$$
$$SL^{-} = \sum_{i=1} \sum_{j=1} S\underline{\delta}_i \delta_j L^{-}{}_{ij}$$

**Single Dot, Product of 2 Tensors**

Definition:

Let $0^{-}, L^{-}$ be tensors. We defined the single dot product of $\underline{0^{-}}$ and $\underline{L^{-}}$

Written $\underline{0^{-}} \bullet \underline{L^{-}}$ so that the IL component of $\underline{0^{-}} \bullet \underline{L}^{-} = \sum 0^{-} ij\ L^{-} jl$

$$L^{-} = \begin{pmatrix} L^{-}{}_{11} & L^{-}{}_{12} & L^{-}{}_{13} \\ L^{-}{}_{21} & L^{-}{}_{22} & L^{-}{}_{23} \\ L^{-}{}_{31} & L^{-}{}_{32} & L^{-}{}_{33} \end{pmatrix}, \quad \underline{0^{-}} \begin{pmatrix} 0^{-}{}_{11} & 0^{-}{}_{12} & 0^{-}{}_{13} \\ 0^{-}{}_{21} & 0^{-}{}_{22} & 0^{-}{}_{23} \\ 0^{-}{}_{31} & 0^{-}{}_{32} & 0^{-}{}_{33} \end{pmatrix}$$

$$\underline{0}^{-} \bullet \underline{L}^{-} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \text{ where } S_{il} = \sum_{j=1}^{3} 0^{-}{}_{ij} L^{-}{}_{jL}$$

$$= 0^{-}{}_{i1}\ L^{-}{}_{1L} + 0^{-}{}_{i2} L^{-}{}_{2L} + 0^{-}{}_{i3}\ L^{-}{}_{jL}$$

**Dot Product of a Tensor and a Vector**

Let $L^-$ be a tensor, let $\underline{v}$ be a vector. The dot product of $\underline{L}^-$ and $\underline{v}$ written $\lfloor \underline{L}^-, \underline{V} \rfloor$ is defined by

$\lfloor \underline{L}^- \bullet \underline{V} \rfloor = \sum_{0} \delta_i \left( \sum L^-{}_{ij} V_j \right)$ so that ith component of $\lfloor \underline{L}^- \bullet \underline{V} \rfloor = \sum \underline{L}^-{}_{ij} V_j$

**Example**

If $\underline{L}^- = \begin{pmatrix} L^-{}_{11} & L^-{}_{12} & L^-{}_{13} \\ L^-{}_{21} & L^-{}_{22} & L^-{}_{23} \\ L^-{}_{31} & L^-{}_{32} & L^-{}_{33} \end{pmatrix}$ & $V = (V_1, V_2, V_3)$

The first component is $\sum_{j=1}^{3} L^-{}_{1j} V_j = L^-{}_{11} V_1 + L^-{}_{12} V_2 + L^-{}_{13} V_3$

The second component is $\sum_{j=1}^{3} L^-{}_{2j} V_j = L^-{}_{21} V_1 + L^-{}_{22} V_2 + L^-{}_{23} V_3$

The third component is $\sum_{j=1}^{3} L^-{}_{3j} V_j = L^-{}_{31} V_1 + L^-{}_{32} V_2 + L^-{}_{33} V_3$

So [ $\underline{L}^- \bullet \underline{V}$ ]= ( $S_1$, $S_2$ $S_3$ )

Let consider

$\lfloor \underline{v} \bullet \underline{L}^{-1} \rfloor = \sum \delta_i V_i L^-{}_{ji} \quad or \quad \sum \delta_i \sum V_j L^-{}_{ji}$

The ith component = $\sum V_j \vec{\underline{L}}^-{}_{ji}$

First component = $\sum_{j=1}^{3} V_j \underline{L}^-{}_{ji} = V_1 L^-_{11} + V_2 L^-_{21} + V_3 L^-_{31}$

Second component = $\sum_{j=1}^{3} V_j \underline{L}^-{}_{i2} = V_1 L^-_{12} + V_2 L^-_{22} + V_3 L^-_{32}$

Third component = $\sum_{j=1}^{3} V_j \underline{L}^-{}_{j3} = V_1 L^-_{13} + V_2 L^-_{23} + V_3 L^-_{33}$

Note $L^-$ is symmetric if $L^-{}_{ij} = L^-{}_{ji}$

Otherwise $L^-$ is not symmetric hence
$\left[\underline{L}^- \bullet \underline{V}\right] \neq \left[\underline{V} \bullet \underline{L}^-\right]$

**Example**

Prove that (i) $[\underline{\delta} \bullet \underline{v}] = [\underline{v} \bullet \underline{\delta}] = \underline{v}$
(ii) $[u\underline{v} \bullet w] = \underline{u}[\underline{v} \bullet w]$

## 3.9 Double Product of 2 Tensors

Let $\delta, L^-$ be tensor
Definition: we define the double product of $0^-$ and $L^-$ written $\left(0^- : L^-\right)$ as
$\left(0^- : L^-\right) = \sum_i \sum_j 0^-_{ij} \underline{L}^-_{jl}$
In the same way
$\left(\underline{L}^- \bullet \underline{u}\underline{w}\right) = \sum_i \sum_j L^-_{ij} U_j W_i$
Similarly
(i) $(\underline{v}\underline{w} : \underline{x}y) = \sum\sum V_i W_j X_j Y_i$
(ii) $\left(\underline{0}^- : L^-\right) = \sum L^-_{ij}$ , $\delta_{ij} = a \ \ if \ \ i \neq j$

Prove

$$0^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - L^- = \begin{pmatrix} L^-_{11} & L^-_{12} & L^-_{13} \\ L^-_{21} & L^-_{22} & L^-_{23} \\ L^-_{31} & L^-_{32} & L^-_{33} \end{pmatrix}$$

$$\left(\underline{0}^- \bullet \underline{L}^-\right) = \sum_{i=1}^{3} \sum_{i=1}^{3} 0^-_{ij} \underline{L}^-_{ji}$$
$$= \sum_{i=1}^{3} \left(0^-_{i1} L^-_{1i} + 0^-_{i2} \underline{L}^-_{2i} + 0^-_{i3} \underline{L}^-_{3i}\right)$$
$$= 0^-_{11} L^-_{11} + 0^-_{22} \underline{L}^-_{22} + 0^-_{33} \underline{L}^-_{33}$$
$$= \int_{11} \underline{L}^-_{11} + \int_{22} \underline{L}^-_{22} + \int_{33} \underline{L}^-_{33} \text{ i.e. } \int_{ij} = 0 \ \ if \ \ 0 \neq j \ \& \delta_{ij} \neq uf \ \delta = j$$
$$\succ \left(\underline{0}^- \bullet \underline{L}^-\right) = \underline{L}^-_{11} + \underline{L}^-_{22} + \underline{L}^-_{33}$$

## 4.0 CONCLUSION

The tensor analysis discussed in this unit is to familiarize you with high language being used in higher mathematics. You need to study this unit properly so that you can able a lot of future mathematics courses.

## 5.0 SUMMARY

In this unit we have learnt about the following:

- Summation Convention in Tensor Analysis,
- Product of Tensors.
- Covariant Tensors , and Contra variant Tensors
- The gradient ,Divergent and Curl of Tensors
- Tensors of various ranks

You are required to master them properly so that you will able to do various exercises and Tutor marked- assignment in this unit.

## 6.0 TUTOR-MARKED ASSIGNMENT

(1) Write each of the following using the summation convention.

(a) $d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \phi}{\partial x^2} dx^2 + .... + \frac{\partial \phi}{\partial x^N} dx^N$.

(2) If $A_k^{pr}$ and $B_s^m$ are tensor. Prove that $C_{qs}^{prm} = A_q^{pr} \; B_s^m$ is also a tensor

$$\overline{A}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^p} \cdot \frac{\partial \overline{x}^j}{\partial x^r} \cdot \frac{\partial x^q}{\partial \overline{x}^k} A_q^{pr}$$

$$\overline{B}_n^m = \frac{\partial \overline{x}^L}{\partial x^m} \cdot \frac{\partial x^s}{\partial \overline{x}^r} B_s^m$$

$$\Rightarrow \overline{A}_k^{ij} \bullet \overline{B}_v^L = \overline{C}_{kr}^{ijL} = \frac{\partial \overline{x}^i}{\partial x^p} \bullet \frac{\partial \overline{x}^j}{\partial x^r} \bullet \frac{\partial x^q}{\partial \overline{x}^k} \bullet \frac{\partial \overline{x}^L}{\partial x^m} \bullet \frac{\partial x^s}{\partial \overline{x}^r} A_{qs}^{prm}$$

(b) $\frac{d\overline{x}^k}{dt} = \frac{\partial \overline{x}^k dx^1}{\partial x dt} + \frac{\partial \overline{x}^k}{\partial x^2} \frac{d}{\;}$

## 70. REFERENCES/FURTHER REDINGS

P.D.S. Verma: Engineering Mathematics
F.B. Hildebrand: Advanced Calculus for Application