MTH 122
INTEGRAL CALCULUS

NATIONAL OPEN UNIVERSITY OF NIGERIA

MTH 122: INTEGRAL CALCULUS

COURSE GUIDE

NATIONAL OPEN UNIVERSITY OF NIGERIA

Acknowledgement: I acknowledge previous writers on their field (some of which are listed for further reading) whose ideas and works have made it possible for me to present my point.

TABLE OF CONTENT

INTRODUCTION

This course is the second of the two calculus courses that you are required to study. In the introduction of calculus, you were informed about the importance of the course calculus to humanity. Also you were informed about those categories of students which need calculus as a working tool for their respective programmes in the National Open University of Nigeria. Therefore, in this introduction, you might not be told what you are already familiar with about the subject calculus. However, for emphasis, the second course of calculus deals with the branch of calculus known as 'Integral Calculus' or 'anti-differential calculus'. Interestingly, this second course in calculus happens to be the first that was known to the early mathematicians in the sense that the study of computation of areas under a curve is older than the study of differential calculus. If you know the distance function of a moving object you can find the velocity by differentiating the distance functions. In this place you will be given the velocity and will be required to find the distance a body travels within a specific interval of time. In other words, integral calculus deals with the inverse operation of differentiation. It extends the concept of addition to enable you to find the sum of a continuously changing quantities then integration continues where finding the sum of changing quantities by discrete method fails. Thus integral calculus is widely used in most situation that requires finding the sum of quantities like computing the areas under curves, volumes generated by rotating certain solids along their axis of symmetry, the length of arc, work done by moving a body along a straight line etc.

In this course, you will study one of the major applications of integral calculus, learn the computation of areas under one or more curves. In unit 1 you will be introduced to the simple numerical method of computing area under a curve. Then in unit 9 the use of integration will be introduced in finding the areas bounded by curves. The relationship between the area under a curve and the definition of definite integration was what lead to the symbol S of integration to be an elongated S, where S stands for a sum. As was done in calculus, I emphasis has been placed on the technique of integration so that you can narrow your guess work appreciably. The solved examples provided throughout in the study units will enable you work through the course effortlessly.

WHAT YOU WILL LEARN IN THE COURSE.

In this course, you will learn how to compute areas under a curve. You learn theorem (without proof) that connects integration and differentiation and called them inverse operations to each other i.e. integration is the inverse operation of differentiation. You will use this theorem which is known as fundamental theorem of integral calculus to develop techniques of indefinite integration. You will learn how to use integration to compute the areas bounded by two curves, the volume generated by solid of revolution, the distance covered by a moving body with a known constant velocity etc.

COURSE AIM

This course aim at developing your skills in the art of integration with a little effort on your part. This is easily achievable by recalling previous knowledge gained from calculus I. Thus in this course, special techniques are introduced that will make integration more of a routine than a guess work.

COURSE OBJECTIVES

On successful completion of this course, you should be able to

i. compute numerically the area under a curve
ii. evaluate definite integrals
iii. evaluate indefinite integrals
iv. evaluate integrals involving trigonometric functions such as sinx, cosx, tan x, colx cosecx and secx.
v. Evaluate integrals involving rational functions involve $a^2 \pm u^2$, $\sqrt{a^2 \pm u^2}$ etc.
vi. evaluate integrals involving rational functions of sinx, cosx, tanx etc.
vii. evaluate integrals involving product functions such as e^{au}Cu2 bu, e^{au} $\int$1-bu $e^x X^n$ etc.

viii. obtain reduction formular for certain categories of functions.
ix. compute area bounded by two intersecting curves.
x. compute the volumes of solid of revolution
xi. find the distance traveled by a moving object with a constant velocity.
xii. compute the work done by compressing or stretching a spring.

COURSE MATERIALS

Herewith list of materials that you will need in order to successfully complete this course:

3 Course Guide for MTH 111
4 Study Units for MTH 111
5 Recommended list of books
6 Assignment file
7 Dates of tutorials, Assessment and Examination

COURSE OVERVIEW

There are 2 modules in this course. Each containing 5 units.

Module I:

Unit 1: Computation of Areas by Calculus
Unit 2: Definite Calculus
Unit 3: Indefinite Integral
Unit 4: Integration of Transcendental functions
Unit 5: Integration of Powers of Trigonometric functions

Module II:

Unit 6: Further Techniques of Integration I
Unit 7: Further Techniques of Integration II
Unit 8: Further Techniques of Integration III
Unit 9: Application of Integration I
Unit 10: Application of Integration II

SET TEXTBOOKS

The following are recommended textbooks you could borrow or purchase them:

1. Godman A, Talbert J.F. Additional Mathematics Pure and Applied in S.I. Longman

(viii) Thomas G.B. and Finney R.L. (1982) "Calculus and Analytic Geometry 5th Ed. Addison – Wesley Publishing Co. World student series Edition, London, Sydney, Tokyo, Manila Reading.

(ix) Satrino LS & Einar H. Calculus 2nd Edition John Wiley & Sons
1. New York London, Sydney, Toronto.

4. Osisiogu U.A, Nwozu C.R. et al (2001) Essential Mathematics for Applied and Management Sciences. Bestsoft Educational Book, Nigeria.

5. Osisiogu U.A. (Ed) (2001) Fundamental of Mathematical Analysis Vol. I, Bestsoft Educational Books, Nigeria.

6. Osisiogu U.A. (Ed) (2001) Fundamental of Mathematical Analysis Vol. II , Bestsoft Educational Books, Nigeria.

ASSIGNMENT FILE AND TUTOR MARKED ASSIGNMENT (TMA)

There are at lest 3 exercise at the end of each section of every unit. These exercises are there as reinforcement to each topic studied. You are to do it all alone. The answers are supplied for self evaluation.

There are two categories of assignment given throughout this course. The first are exercises given at the end of each section of every unit. These exercises are given to reinforce your understanding of the concept studied. You are required to do it all alone. The answers to the exercises are given in the unit for self evaluation. This will make you monitor your progress.

The second are exercises given at the end of each unit. There are at least ten of them. These are assignment that you must do and submit to your tutor at your study centre. These assignments will be supplied to you in your assignment files.

EXAMINATION and MARKING

The final examination for the course MAT 124 will be for a duration of 2 hours.

STRATEGIES FOR STUDYING THE COURSE

Integration involves finding a function whose denuative is the integrand. As such it involves certain level of guess work. In view of this, you are required to spend much time reviewing your differential calculus and trigonometry functions. Thus differential calculus is a necessary tool you need to develop your skills in techniques of integration. Therefore, you are required to spend 4 hours or more studying each unit. Because of the time required to study a unit, this course has been designed to contain only 10 units instead of 20 units. You are required to spend about one hour or more on the solved examples of each unit. By doing this, you would have developed enough skills that most of your guesses will be accurate. While reading through this course, make sure that you check up any topic you are referred to in any previous unit you have studied.

SUMMARY

In this course, you have studied how to:

1. Compute area under a curve as a sum of areas of rectangles inscribed or circumscribed under a curve within a given interval.
2. Apply the fundamental theorem of integral calculus in evaluating the definite an indefinite integrals.
3. To evaluate integrals using both standard notation an properties of indefinite integrals.
4. Derive standard integration formulas such as $\int \sin u\,du = -\cos u + c$, $\int \frac{1}{u} du = \ln|u| + C$ etc.
5. Evaluate integrals of odd and even powers of cosx and sinx.
6. Obtain reduction formula for $\int e^{ax}$ Cu-bv dx, $\int e^x x^n dx$ $\int \cos^n x\,du$ etc.
7. Evaluate integrals of this type

   $\int \frac{du}{a^x+u^2}$, $\int \frac{du}{\sqrt{a^2 \pm u^2}}$ etc

8. Compute integrals by the following methods

   (18) partial fractions
   (19) completing the square
   (20) half angles formula and
   (21) method of integration by parts.

9. Use definite integration in finding

   (1) areas under two curves
   (2) distance traveled by an object with a constant velocity
   (3) volumes of solid of revolution etc.

MTH 122: INTEGRAL CALCULUS

COURSE DEVELOPMENT

Course Developer
Dr. U. A. Osisiogu
Ebonyi State University
Abakaliki

Unit Writer
Dr. U. A. Osisiogu
Ebonyi State University
Abakaliki

Programme Leader
Dr. Makanjuola Oki

Course Coordinator
B. Abiola

NATIONAL OPEN UNIVERSITY OF NIGERIA

CALCULUS II

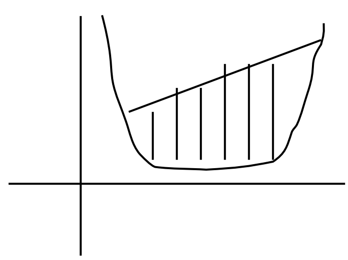
UNIT 1

COMPUTAION OF AREAS BY CALCULUS

TABLE OF CONTENT

1.0 INTRODUCTION

One of the early mathematicians that attempted to find the area under a curve was a Greek named Archimedes. He used ingenious methods to compute the area bounded by a parabola and a chord. See Fig (1.1).



In this unit, you will study how to develop necessary tools of calculus to compute areas under curve as a mere routine exercise. The area under a curve gave birth to the second branch of calculus known as integration. The tools that will be developed here will naturally lead to the definition of integration in the next unit – unit 2. Recall that the word to integrate connotes "whole of" which could be interpreted to mean "find the whole area of". This concept is what will be introduced in this unit and this will be fully developed in the next unit.

2.0 OBJECTIVES

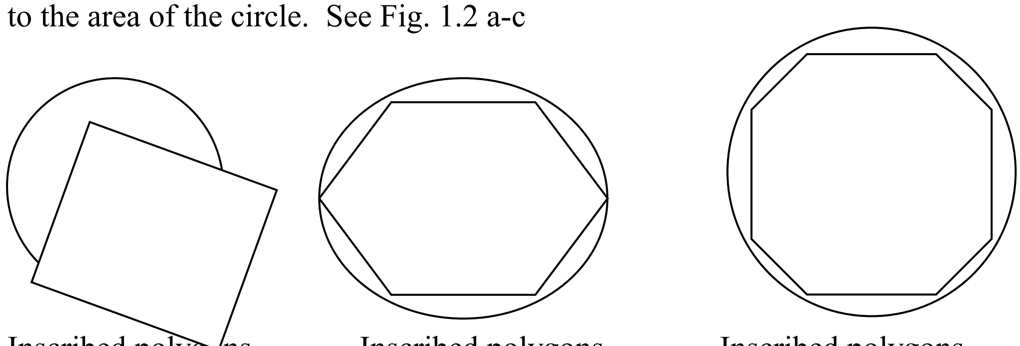
After studying this unit, you should be able to correctly

i. approximate area under a curve by the sum of areas of rectangles inscribed in the curve.
ii. approximate the area under a curve by the sum of the areas of rectangles circumscribed over the curve.
iii. define a partition of a closed interval (a, b)
iv. compute the exact value of the area under a curve by the limiting process.

3.1 AREA UNDER A CURVE

You are quite familiar with the computation of the areas of plane figures such as triangles, parallelogram trapezium, regular polygons etc. Interestingly, you studied in elementary geometry that the area of a regular polygon can be computed by cutting it into triangles and sum up the areas of the triangles.

You are also aware t hat the area of a circle is π^{r2}. This formula was derived by the method of limit. You could recall that the limit of the areas of inscribed regular polygons as the number of sides approaches infinity is equal to the area of the circle. See Fig. 1.2 a-c



Inscribed polygons of 4 sides — Inscribed polygons of 6 sides — Inscribed polygons of 8 sides

Fig. 1.2a **Fig. 1.2b** **Fig. 1.2c**

Let y = f(x) be a continuous function (see the first course on calculus i.e. calculus I unit 4 for definition of continuous function) of x on a closed interval [a, b]. In this case for better understanding, you assume that the f(x) is positive in the closed interval i.e. $f(x) \geq 0$. for all XE [a, b]. Then the problem to be considered is to calculate the area bounded by the graph y = f(x) and the vertical lines y = f(a) and y = f(b) and below by the x – axis as shown in Fig. 1.3.

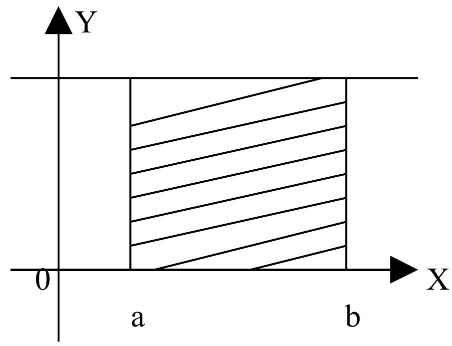


Fig. 1.3

You can start by dividing the area into n thin strips of uniform width $\Delta x = \frac{(b-a)}{n}$ by lines perpendicular to the x – axis at the end points x = a and x = b and many intermediate points which can be numbered as X_1, X_2, X_{n-1} (see fig 1.4).

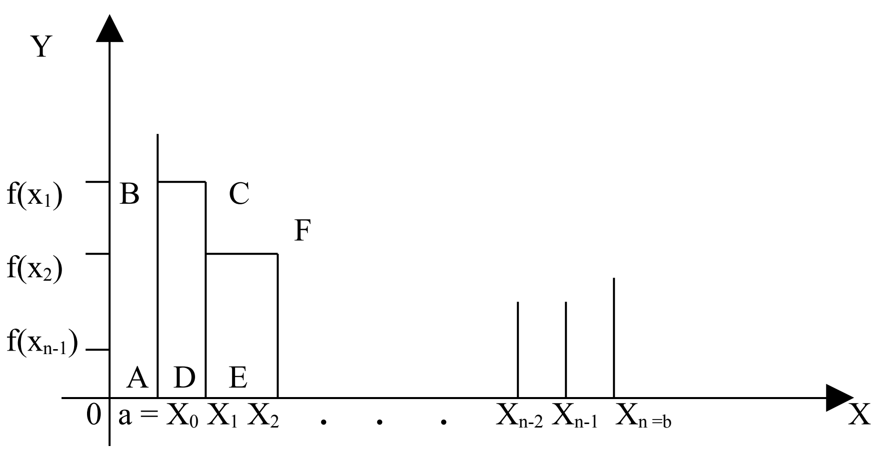


Fig. 1.4

The sum of the areas of these n rectangular strips gives an approximate value for the area under the curve. To put the above more mathematically, you can define the area of each strip in terms of f(x) and x. Given that $\Delta x = x_1 - a =$

x_2-x_1 = .. = b-x_{n-1}. For example the area of the rectangular strip ABCD in Fig. 1.4 above is given as:

$$\text{Area of ABCD} = f(x_2) . (x_1-x_0) = f(x_2)\ \Delta x$$

Example:

Suppose $f(x) = x^2-3$ in Fig 1.5 with n = 6 were a = 2, b = 8,
dx = 8-2 = 6
Therefore: dx = 1 i.e. you have 6 rectangular strips.

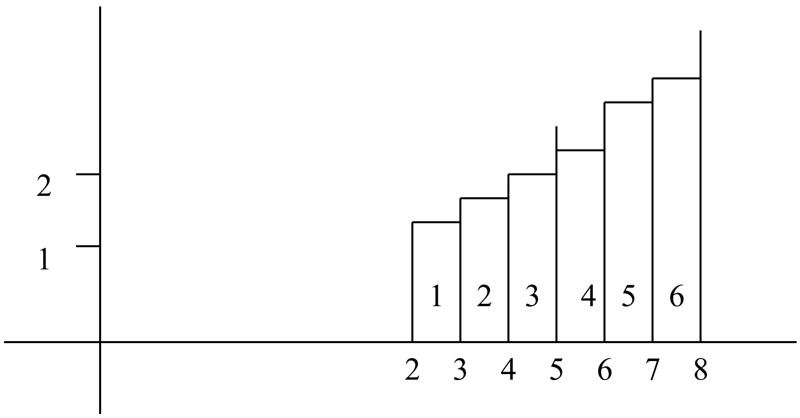


Fig. 1.5

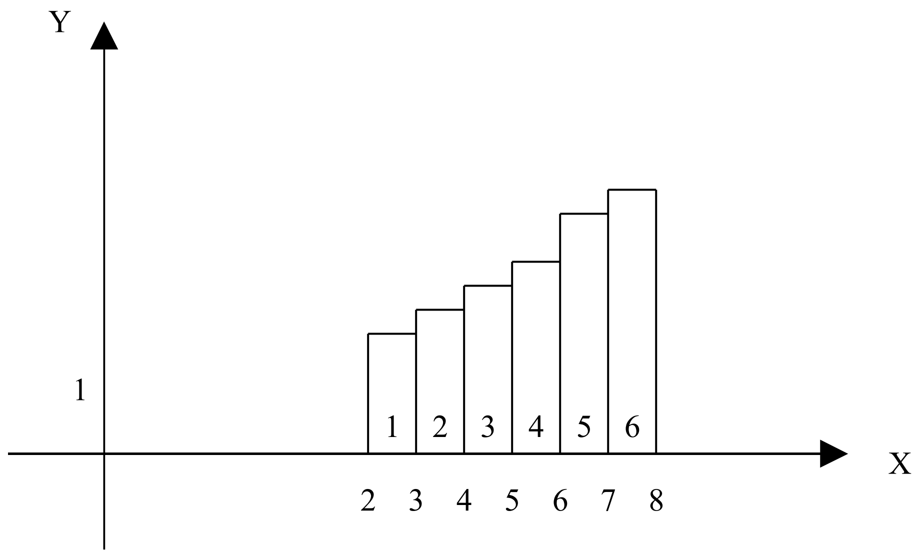
Area is given as sum of

$f(2)\ \Delta x$	=	1.1	=	1
$f(3)\ \Delta x$	=	6.1	=	6
$f(4)\ \Delta x$	=	13.1	=	13
$f(5)\ \Delta x$	=	22.1	=	22
$f(6)\ \Delta x$	=	33.1	=	33
$f(7)\ \Delta x$	=	46.1	=	46

In fig. 1.4 above the area under the curve is larger than the sum of the areas of the inscribed rectangular strips numbered 1 to 6 i.e. sum of areas of strips = 1+6+13+22+33+46 = 121 which is less than area under curve.

Example:

Using the same example $Y = x^2-3$ use circumscribed rectangular strips instead of inscribed ones to compute the area under the curve. See Fig. 1.6



Area is given as the sum of

f(x) . Δx	=	6.1	=	6
f(x) . Δx	=	13.1	=	13
f(x) . Δx	=	22.1	=	22
f(x) . Δx	=	33.1	=	33
f(x) . Δx	=	46.1	=	46
f(x) . Δx	=	61.1	=	61

Area = 6+13+22+33+46+61 = 181

As should be expected this area is greater than the area under the curve $f(x) = x^2-3$.

In the computation with the circumscribed rectangular strips the sides of the rectangles are assumed in this case to be the points of the function in their respective subintervals. In the case of the inscribed rectangles, the sides of the rectangles are the minimum values of the function in their respective subintervals.

Therefore the area under the curve lies between the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribe rectangles. This takes to the issue of limit. Therefore it will be right to say as $n \rightarrow \alpha$ $\Delta x \rightarrow 0$ this implies that the

$\text{Lim (Max Area – Min Area)} = 0 \text{ as } \Delta x \longrightarrow 0.$

From the foregoing, you can now define the area under curve as the limit of the sums of the areas of inscribed (circumscribed) rectangles as their common base of length dx approaches zero and the number of rectangles increases without bound. In symbols you can write the above limit as:

$$A = \lim [f(x_0)\,\Delta x + f(x_7)\,\Delta x + .. + f(x_{n-1})]\,\Delta x \quad n \longrightarrow \infty$$

$$= \lim [f(x_1)\,\Delta x + f(x_2)\,f + .. + f(x_{n-1})]\,\Delta x \quad n \longrightarrow \infty$$

OR

$$A = \lim_{n \to x_1} \sum_{K=10}^{n-1} f(x_k)\,dx = \lim_{n \to x_1} \sum_{K=1}^{n} (fx_k)\,dx$$

Exercise:

Repeat the above example using n = 10. Find the difference between the sum of areas of the inscribed rectangle (i.e. the minimum area) and the sum of areas of the circumscribed rectangles (i.e. the maximum area).

3.2 PARTITION OF A CLOSED INTERVAL

Let [a, b] be a bounded closed interval of real numbers. A partition of a closed interval [a, b] is a finite set of points

$P = \{a = x_0, x_1, x_2, .., x_{n-1}, x_{n-b}\}$ where

$a = x_0 < x_1 < x_2 < .. < x_{n-1}, x_n = b$

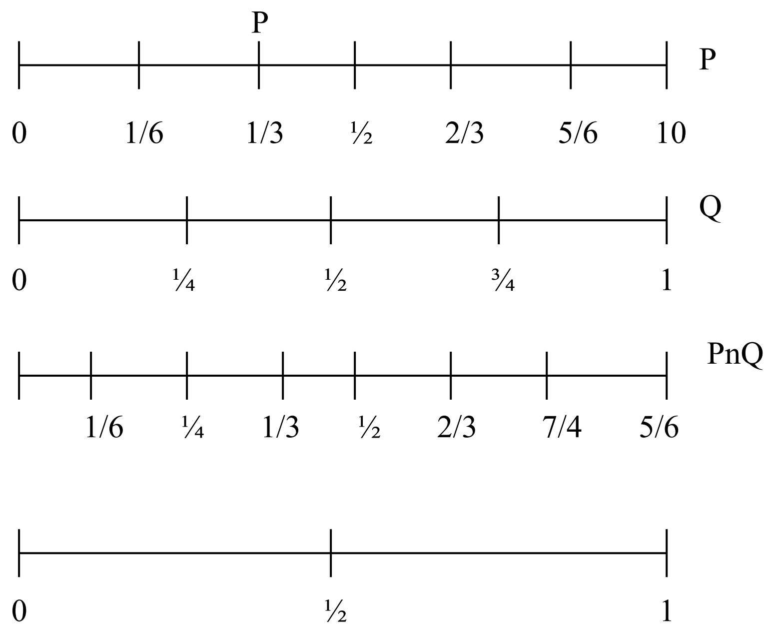
Example:

P = {0, 1/6, 1/3, ½, 2/3, 5/6, 1} and
Q = {0, ¼, ½, ¾, 13 are both partition of {0, 13}

PUQ = {0, 1/6, ¼ , 1/3, ½ , 2/3, ¾, 5/6 1} is a partition of [0, 1]

PnQ = {0, ½, 1} is a partition of [0,1]

See fig. 10.6(a) to (c)



A partition of P = $\{x_{0,}\ x_{1,}\ ..,\ x_n\}$ of [a, b] divides [a, b] into n closed sub interval $[x_{0,}\ x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, X_n]$

The closed interval

$[x_{r-1}, x_r]$ is called the r^{th} subinterval of the partition.

Given a partition of P[a = x_0, x_2, …, x_n = b] the length of the subinterval s are the same and it is denoted by $\Delta x_r = x_r - x_{r-1}$

This equal to the length of the interval [a, b] divided by the number of subintervals n

i.e. $\Delta x_r = \frac{b - a}{n}$

Example: Δx for p is $\frac{1 - 0}{6} = 1/6$

Δx for Q is $\frac{1 - 0}{4} = ¼$

Not in all case you will get subintervals of the same length. Example is PUQ
The length of $x_1 - X_0 = 1/6 - 0 = 1/6$
The length of $x_2 - X_1 = ¼ - 1/6 = 1/12$

Such partitions in which the subintervals are not of the same length are called irregular partition.

Exercise: Write down a regular partition for

(1) [2, 8], n = 12
(2) [1, 8], n = 7

Ans:

(i) [2, 5/2, 6/2, 7/2, 8/2, 9/2, 10/2, 11/2, 12/2, 13/2, 14/2, 15/2 16/2]

(ii) [1, 2, 3, 4, 5, 6, 7, 8]

3.3 COMPUTATION OF AREAS AS LIMITS

In this section you will combine the results of section 3.1 and 3.2 to compute the areas under curves using the limiting process.

Example

A good starting point is to consider the area under the curve Y = X (see Fig. 10.7)

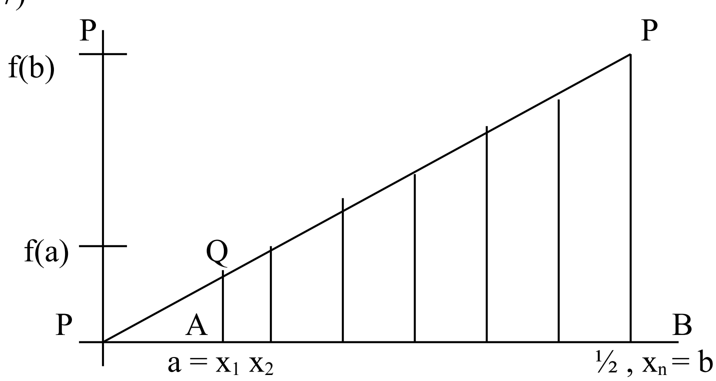


Fig 10.7

which the interval Xϵ [a, b] let there be n-regular partition of [a, b] i.e.

$$\Delta x = \frac{b - a}{n}$$

$P\ [a, x_1, x_2, .., x_{n-1}, x_n = b]$

$x_1 = a + xx$

$x_2 = a + 2\Delta x$
$x_3 = a + 3\Delta x$

$x_{n-1} = a + (n-1)\ \Delta x$

Areas of inscribed rectangles are

$f(a)\ .\ \Delta x = a.\Delta x$
$f(x_1)\ .\ \Delta x = (a+\Delta x).\Delta x$
$f(x_2)\ .\ \Delta x = (a+2\Delta x).\ \Delta x$
..
$f(x_{n-1})\ \Delta x = (a+(n-1)\Delta x)\ .\ \Delta x$

Sum of the areas of the rectangles is given as

$\surd = (a\ .\ \Delta x + (a + \Delta x) + \ldots + (a + (n-1)\ \Delta x).\Delta x$
$= [a+((1+2+3+\ldots(n-1)\ dx$

$= na + (\sum_{K=1}^{n-1}, K)\ \Delta x\ )\ \Delta x$

$(\sum^{n-1} K = \frac{(n-1)n}{2}$ (The sum of an arithmetic 1 progression with different d=1)

$S= [\ na + \frac{(n-1)n}{2}\ \ d\underline{x}\ ]\ dx$

but $dx = \frac{b-a}{n}$ therefore

$S = [na + \frac{(n-1)n}{2}\ \frac{b-c)}{n}\ ]\ \frac{b-c}{n}$

$= [a + \frac{b-a}{2}\ \frac{n-1}{n}\ ]\ (b-a)$

Taking limit as $n \to \infty$ you get

$$\text{Lim } S = \lim_{n \to \infty} \lim_{n \to \infty} \left(a + \frac{b-a}{2}\ .\ \frac{n-1}{n}\ (b-a) \right)$$

$$= (a + \frac{b-a)}{2}\ (b-a)\ \text{lims}_{n\ \infty \to}\ \frac{n-1}{n}$$

$$= (a + (\underline{b-a})\ (b-a\ )\ .\ 1$$

2

$$= \frac{a + b}{2} . (b - a)$$

In fig. 10.8, the area of trapezium AQPB is the same as the area under the curve and as you know the area of trapezium is given as:

½ base x sum of two parallel sides
= ½ (b – a) x f(a) + f(b)
= ½ (b – c) (b + a)

Example

Find the area under the graph Y = x + 1 $0 \leq x \leq 6$

Solution:

Let n be a positive integer that there be a partition of [a, b] into n regular partition.

Therefore; $\Delta x = \frac{b}{n}$

$x_1 = \Delta x$
$x_2 = 2\Delta x$
$x_3 = 3\Delta x$
..
$x_{n-1} = (n-1)\ \Delta x$

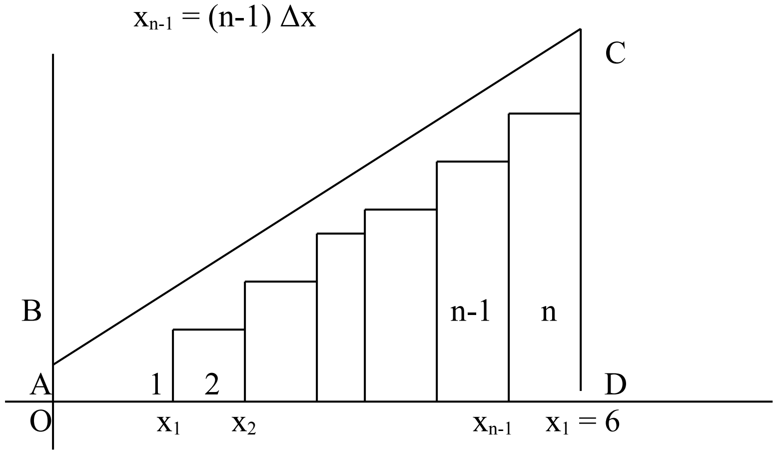


Fig. 10.8

Area of (n-1) rectangles is given as

$f(0).\Delta x \quad = \quad 1.\Delta x$

$f(x_1).\Delta x = (\Delta x + .1)\,\Delta x$

sum of areas of rectangles is

$$S = \Delta x + (\Delta x+1)\Delta x + (2\Delta x+1)\,\Delta x + (3\Delta r+1)\Delta x + \ldots + (n-1)\,\Delta x + 1)\;\Delta x$$
$$= (\Delta x + (\Delta x+1) + (2\Delta x+1) + : (n-1)(\Delta x+)\,\Delta x$$
$$= [\Delta x + n +)\sum_{K=1}^{n-1} k\Delta x]\,\Delta x$$

$S = [\Delta x + (n + \frac{(n-1)n}{2}\,\Delta x]\,\Delta x$ let $\Delta x = \frac{1}{n}$

then

$$S = \left[\frac{b}{n}\right] + \left[\left(n + \frac{(n-1)}{2}\right) n\,\frac{b}{n}\right)\right]\frac{b}{n}$$

$$= \frac{b}{n} + \left(1 + \frac{b}{n} . \frac{n-1)}{2}\right) . b$$

Taking limits as $n \to \infty$

$$\text{Lim } S = \lim_{n \to \infty} \frac{b}{n} + b \lim_{n-1\,\infty} \left(1 + \frac{b}{2}\,\frac{(n-1)}{n}\right)_{n \to \infty}$$

$= 0 + b\,(1 + b/2)$

$$\lim_{n \to \infty} S = b + \frac{b^2}{2}$$

Exercise:

Show that the area of the trapezium ABCD in Fig. 10.8 is equal to $\frac{b(b+2)}{2}$

4.0 CONCLUSION

In this unit, you have studied how to find an approximate value of the area under a curve by computing the sums of areas of rectangles inscribed under the cure and circumscribed over if you have defined a partition of a closed interval. You have studied that as the number of partitions of a closed interval [a, b] is increased without bound the value of the sum of the areas of the rectangles (inscribed or circumscribed) approaches the exact value of the area under the curve in the given interval [a, b] that is the limit of the sum of areas of the rectangles is equal to the exact area under the given curve as the number n of partition tends to infinity or the length dx of the subinterval of the partition tends to zero.

5.0 SUMMARY

In this unit you studied how to

- compute the minimum value of area under a curve i.e. sum of area of rectangles inscribed under a curve within an interval
- compute the maximum value of the area under a curve i.e. the sum of areas of rectangles circumscribed over the curve.
- define a partition of a closed interval [a, b] i.e. $a= x_1 < x_2 < \ldots < x_2 = b$ = P[a, b]
- compute the exact area under a curve in a given interval [a, b] by taking the limit of the sum of the areas of the rectangles (inscribed or circumscribed) as the number n of partition of [a, b] is increased without bound i.e. $A = \lim_{n \to \infty} \int_n$ where $dx = \frac{b - a}{n}$

6.0 EXERCISE

1. Show that the sets

   {0, 1} {0, ½, 1}, {0, ¼, ½, 1} and
   {0, 1/4 , 1/3, ½, 5/8, 1} are partition of {0, 1}

2. Which of the partition of [0, 1] in exercise (1) above are regular?

3. Find the minimum and maximum values of the area under the curve $f(x) = 2x$ for $x \in [0, 1]$ and P(0, ¼, ½, 1)

2. Find the minimum value of the area under the curve $f(x)= 1- x$ $x \in [0,2]$ P($0_1$1/3, ¾, 1, 2).

3. Find the area under the curve $Y = x^2$ $X \in [0, b]$ by taking appropriate limits.

4. Find the area under the curve $Y = mx$ $a \leq x \leq b$ by taking appropriate limits.

5. Sketch the graph of Y = X + 1. Divide the internal into n = 6 subintervals with d<u>x</u> = (b – a)/6. Sketch the inscribed rectangles.

6. Repeat ∑ x 7 but this time sketch the circumscribed rectangle.

7. Compute the sums of areas in Є x 7 and Є x 8 above.

8. Find the area under the curve Y = x + 1 a ≤ b by taking appropriate limits of results of exercise 9 above.

UNIT 2

DEFINITE INTEGRAL

TABLE OF CONTENT

1.0 INTRODUCTION

In unit 1, you studied how to compute the area under a curve and showed how you could estimate it by computing sums of area of rectangles. Using the above estimate you applied the concept of limit to get the exact value of the area under a curve. These methods were applied to functions or graphs that could easily be sketched i.e. not too complicated functions. In this unit, you will be introduced to the famous path taken by Leibniz and Newton in showing how exact areas can be computed easily by using integral calculus. It is necessary you refer once more to unit 1 of this course before embarking on this one. It will help you have a proper grasp of this unit if you do so.

2.0 OBJECTIVES

After studying this unit you should be able to:

- Define the definite integral of a function within an interval [a, b].
- Evaluate definite integrals of function.
- State the fundamental theorem of integral calculus.

3.0 DEFINITION OF DEFINITE INTEGRAL

In unit 1, you studied that the sum of the areas of inscribed rectangles gives a lower (minimum) approximation of the area under the curve of the function f(x). If you list all the values of the function f(x) in a given interval [a, b] and take the least among these value you will have what is known as the infimum of f(x) for all x€ [a, b]

i.e. Inf f(x) X∈[a, b].
let Inf f(x) = M_r and X∈ P[a, b]
when $dx_r = x_r - x_{r-1}$. then the area is $M_r.\ dx_r$. The sum of such area is
$A_L = \sum M_r (x_r – x_{r-1})$ is called the Lower Sum of the function f(x).

If you take the maximum value of f(x) within [a, b] and find their areas i.e. $M_r = \int$up f(x) X∈ $[x_{r-1}\ x_r]$ then the Upper sum for the areas is given as

$$A_u = \sum_{r=1} M_r(x_r – x_{r-1})$$

No known concept has been introduced. You are rewriting sum of areas of a rectangles inscribed under the curve f(x) as $A_L = \sum M_r (x_r – x_{r-1})$ and the sum of areas of rectangles circumscribed over f(x) as

$$A_u = \sum M_r\ x_r – x_{r-1})$$

Once you keep the fact you will not run into any difficulty understanding what follow next.

Definition: The unique number I which satisfies the inequality

$A_L(P) \leq I \leq A_u(P)$ for all partitions P of [a, b] is called the definite integral (or more simply the integral of f on [a, b] and is written as:

$$I = \int_a^b f(x)dx$$

This symbol S dates back to Leibniz and it is called he integral sign. It is an elongated S, which represents sum. The numbers in this case are called the limits of integration. The expression $\int_a^b$ f(x) dx (read integrating from b to a with respect to x)

In the above definition, it has been assumed that f(x) is continuous in the closed [a, b]. This condition guarantee the existence of a number I such that

$$A_L(p) \leq I \leq A_L(p)$$

The prove of the above theorem could be found in the text suggested for further reading given at the end of this course.

If $f(x) \geq 0\ \forall\ X \in [a, b]$ then

$$I = \int^b f(x)dx = \text{Area under the curve } f(x)$$

a

Example

Given that $f(x) = K \ \forall \ X \in [a, b]$ show that $\int_a^b f(x)dx = K(b - a)$

Solution: Let $P = \{u, x_o, x_1, x_2, .., x_n = b\}$ be any partition of [a, b]

Since $f(x) = K \ \forall \ X \in [a, b]$ the $f(x_0) = f(x_1) = \ldots = f(x_n)$

Let $A_L(P) = \sum m\Delta X_r = K\Delta X_1 + K + \Delta X_2 + ..K\Delta X_n$

$= K(\Delta X_1 + ... + \Delta X_n) = K(b-a)$

Also

$A_u(P) = \sum M_r\Delta X_r = K(b-a)$

But

$A_L(P) \leq \int_a^b f(x)dx \leq A_u(P)$

then $K(b-a) \leq \int_a^b f(x)dx \leq K(b-a)$

$\Longrightarrow \int^b f(x)dx = K(b-a)$

Example

Given that $f(x) = x$ show that $\int_a^b xdx = \frac{1}{2}(b^2-a^2)$

Solution: Let $P\{ax_{o,} x_1, .. , x_n, = b\}$ be an arbitrary partition of [a, b].

$f(x) = x$ for $X \in [X_r, X_{r2,}]$ for all such subintervals.

So $M_r \leq f(x)\ m_r\ X \in [x_r, x_{rt1}]$ such m_r and m_r exist for each subinterval.

Let $M_r = x_r$ and $m_r = x_{r-1}$

then

$$A_u(P) = \sum_{r=1}^{n} m_r\Delta x_r = \sum_{r=1}^{n} x_r\Delta x_r$$

$$= x_1(x_1-x_o)+x_2(x_2-x_1)+\ldots+ x_n(x_n-x_{r-1})$$

and

$$A_L(P) = \sum_{r-1}^{n} m_r\Delta x_r = \sum_{r-1} x_{r-1}\Delta x_r$$

$$= x_o(x_1-x_o) + x_1(x_2-x_1) + .. x_{2-1}(x-x_{n-1})$$

For each index,

$$x_{r-1} \leq \frac{1}{2}(x_r + x_{r-1}) \leq x_r$$

Therefore

$A_L(P) \leq ½\,(x_1+x_o)\,(x_1-x_o) + ½\,(x_2+x_1)\,(x_2-x_1) + \ldots + ½\,(xn+xn_{-1})$
(x_n-x_{r-1})
$\leq A_u(P)$
but $½\,(x_1+x_o)\,(x_1-x_o) + ½\,(x_2+x_1)\,(x_2-x_1) + .. + ½\,(xn+xn_{-1})$
$(Xn+Xn_{-1})$
$= ½\,(x^2_1 - x^2_0 + x^2_2 - x^2_1 + \ldots + x^2 - x^2n_{-1}) = ½\,(x^{2n} - x^2o)$

$\Longrightarrow A_L(P) \leq ½\,(x^2_n - x^2_o) \leq A_u(P)$

$\Longrightarrow A_L(P) \leq ½\,(b^2-a^2)$

$\Longrightarrow \int_a^b x\,dx = ½\,(b^2 - a^2)$

The following properties of definite integral are hereby stated without their proofs are beyond the scope this course:

1. If $a < c < b$ then $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

2. If $a < b$ then $-\int_a^b f(x)dx = \int_b^a f(x)dx$

3. $\int_a^a f(x)\,dx = 0$

Example: Given that

$\int_0^1 f(x)dx = 6,\ \int_1^3 f(x)dx = 5$

$\int_3^7 f(x)dx = 2$

Find (i) $\int_0^7 f(x)dx$ (ii) $\int_1^3 f(x)dx$ (iii) $\int_1^1 f(x)dx$ (iv) $\int_7^1 f(x)dx$

Solution: (i) $\int_0^7 f(x)dx$

$\int_0^7 f(x)dx = \int_0^t f(x)dx + \int_x^7 f(x)dx$

let t = 3 i.e. $\int_0^7 f(x)dx = \int_0^3 f(x)dx + \int_3^{7f}(x)dx = 5 + 2 = 7$

(ii) $\int_1^3 f(x)dx = \int_1^t f(x)dx + \int_t^3 (x)dx$

let t = 0 $\int_1^3 f(x)dx \int_1^0 f(x)dx + \int_0^3 f(x)dx = -6 + 5 = 1$

(iii) $\int_1^1 f(x)dx = 0$

(iv) $\int_7^1 f(x)dx = -\int_1^7 f(x)dx = [\int_1^3 f(x)dx + \int_3^7 f(x)dx = -[-1+2]$

Exercise: Given that $\int_0^2 f(x)dx = 2$, $\int_0^3 f(x)dx = 4$ and $\int_2^4 f(x)dx = 7$

Find (i) $\int_0^4 f(x)dx$ (ii) $\int_4^3 f(x)dx$ (iii) $\int_3^2 f(x)dx$

(iv) $\int_2^3 f(x)dx$

Answer: (i) 9 (ii) -5 (iii) -2 (iv) 2

3.2 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

To find the value of the function $F(x) = \int_a^b f(x)dt$ for some simple function it could easily be evaluated. Either by the limiting process discussed in unit or by direct evaluation as was done in the previous section. Such process might prove very laborious for certain classes of functions. In this section you will examine the direct connection between differential calculus and integral calculus. This connection was made possible by looking at the summation process of finding areas and volumes and the differentiation process of finding the scope of a target to a curve. It is quite interesting that the process of carrying out inverse differentiation yields an easy tool of solving the summation problem.

So you will now discuss the proof of the fundamental theorem concept behind the theorem is that before you can evaluate a definite integral $S_a^b f(x)dx$ you will first of all find a function Fl(x) whose derivative is f(x). i.e. $F^1(x) = f(x) \ \forall \ X \in (A, B)$. You will now as first step study the proof of the following theorem:

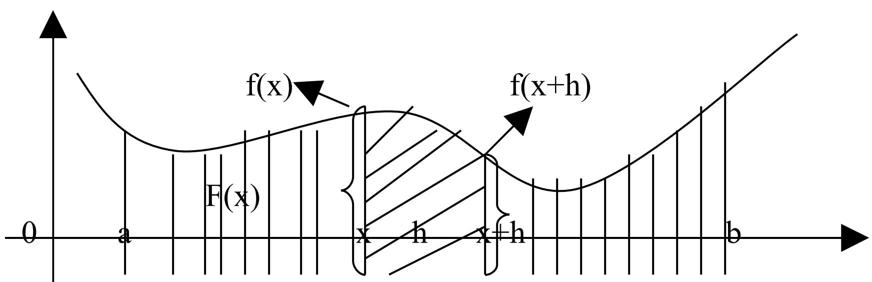
Theorem 1: If f(x) is a continuous function on [a, b], the function F(x) defined on [a, b] by setting $F(x) = \int_a^x f(t)\,dt$

is a(i) continuous function on [a, b] and (ii) satisfies $F^1(x) = f(x)$ for all x in (a, b).

Proof: You will begin with x€ [a, b] and show that

$$\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$$

In figure 2.1 F(x +h) = area from a to x + h



F(x) = area from a to x

F(x + h) - F(x) = area from x to x + h

(Area = base x height)

$$\frac{F(x + h) - F(x)}{h} = \frac{\text{area from x to x + h}}{h} \simeq f(x) \text{ if } h \to 0$$

(note f(x) = height of the area under curve in Fig. 2.1

If $x < x + h \leq b$ then

$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$

(since from statement of theorem $F(x) = \int_a^x f(t)dt$)

It follows therefore that

$$F(x+h) - F(x) = \int_a^{x+h} (f(t)dt + \int_x^a f(t)dt$$

$$= \int_x^{x+h} (f(t)dt$$

Let M_h = maximum value of f(x) on [x, x+h]

and m_h = minimum value of f(x) on [x, x+h] since $M_h\,(x+h-x) = M_h.h$

and $m_h\,(x+h-x) = m_h.h$

therefore, M_h = upper sum (see UNIT 1)
and m_h = lower sum (see UNIT 1)
therefore

$$m_h.h \le \int_x^{x+h} f(t)dt \le M_h.h$$

$$= \quad m_h.h \le \frac{F(x+h - F(x)}{h} \le M_h.h$$

since f(x) is a continuous function on [x, x + h] therefore

$$\lim_{h \to 0^-} m_h.h = f(x) = \lim_{h \to 0^-} M_h.h$$

thus $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$. —— I

In a similar manner you can show that if X Є (a, b), then

$$\lim_{h\ 0^+} \frac{F(x+h) - F(x)}{h} = f(x) \quad \text{——} \quad I$$

Now if xЄ (a, b) then equation (I) and (II) hold

Thus $\lim_{H \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$

and $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F^1(x)$

therefore $F^1(x) = f(x)$

since $F^1(x)$ exists then F(x) must be continuous on (a, b). Before you prove the fundamental theorem of calculus. Look at this definition.

Definition:

A function F(x) is called an anti-derivative for f(x) on (a. b) if and only if

(i) F(x) is continuous on (a, b) and
(ii) $F^1(x) = f(x)$ for all XЄ (a, b)

Using the above definition you can rewrite theorem 1 as

If f is continuous on (a, b) then

$F(x) = \int^{x} f(t)dt$

The above now says to you that you can construct or find an anti-derivative for f(x) by integration f(x). The next theorem you are going to study will tell you that you can evaluate the definite integral $\int_{a}^{h} f(x)dx$ by finding an anti-derivative for f(x).

The Fundamental Theorem of Integral Calculus:
Let f(x) be continuous for all x∈ (a, b) If P(x) is an anti-derivative of f(x) for all x∈ (a, b) then

$\int_{a}^{a} f(x)dx = P(b) – P(a)$

Proof: In theorem 1, the function $F(x) = \int_{a}^{x} f(t)dt$

is an anti-derivative for f(x) for all X∈ (a, b).

If P(x) is another anti-derivative for f(x) for all x∈ (a, b), then it implies that both P(x) and F(x) are continuous for all X∈ (a, b) and also will satisfy that $P^1(x) = F^1(x)$ for all X∈ (a, b). There exist a constant C such that

$F(x) – P(x) = C$

Since $F^1(x) = P^1(x)$ and derivative of a constant is zero
i.e. $F(x) – P(x) = C \Rightarrow F^1(x) – P^1(x) = 0$

Since F(a) = 0 then P(a) + C = 0 and C = -P(a)

This implies that

$F(x) = P(x) – P(a)$ for all X∈ (a, b)

Thus $F(h) = P(b – P(a)$ (x = b)

Since $F(b) = \int_{a}^{b} f(t)dt = P(b)-P(a)$

which is the required result.

3.3 EVALUATION OF DEFINITE INTEGRAL

You are now set to seek or construct anti-derivates F(x) which will evaluate the definite integral given as $\int^{b} f(x)dx$

a

Example: Find $\int_a^b x\,dx$

Solution

Let $F(x) = ½x^2$ as an anti-derivative

Then $\int_a^b x\,dx = ½\,(b^2 – a^2)$

Find the $\int_a^b x^n dx$ when n is a positive integral the anti-derivative to use is

$F(x)= \frac{1}{n+1}\, x^{n+1}$

$\Longrightarrow F^1(x) = x^n \Longrightarrow \int_a^b x^n = F(b) – F(a)$

$= \frac{1}{n+1}\,(b^{n+1} – a^{a+1})$

Notation:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) – F(a)$$

thus $\int_a^b x^4 dx\ [\ \frac{1}{4+1} x^{4+1}]_a^b = 1/5\ (b^5 – a^5)$

Example:

$$\int^2 (6x^5-2x^3-x)dx$$

Let $F(x) = x^6 – \frac{2x^4}{4} – \frac{x^2}{2}$

then $\int_1^2 (6x^5 – 2x^3 – x)dx = [x^6 – \frac{1}{2}x^4 – \frac{x^2}{2}]_1^4$

$= 3960.$

Example:

Evaluate the following integrals by applying the fundamental theorem.

(i) $\int^0 (x-1)\,(x-2)dx$

(ii) $\int_3^7 \frac{dx}{(x-2)^2}$

(iii) $\int_0^1 (x^{3/4} + ½ x ½)\, dx$

(iv) $\int_a^9 (a^2 x - x^4)dx$

(v) $\int_1^3 \frac{2 - x}{x^3} dx$

(vi) $\int_1^8 (\sqrt{t} - \frac{1}{t^2})dt$

(vii) $\int_1^3 \underline{6 - t}dt$

(viii) $\int_1^2 x^2(x-1)dx$

(ix) $\int_1^4 \sqrt{x+1}dx$

(x) $\int_0^1 (x-1)^{17}dx$

Solution: To evaluate $\int_{-3}^0 (x-1)\ (x-1)dx$

you expand the function (x - 1) (x -1)

$= x^2 - 2x + 1$

(i) $\int_0^3 (x-1)\ (x-1)dx = \int_0^3 (x^2-2x+1)\ dx$

let $F(x) = 1/3\ x^3 - x^2 + x$ serve as anti-derivative

therefore $\int_0^3 (x^2-2x+1) = [1/3x^3 - x^2 + x]_0^3 = 3$

(ii) $\int_3^7 \frac{dx}{(x-2)}$

construct a function with derivative as $\frac{1}{(x-2)^2}$ it is not difficult to see that

$$\frac{d}{dx} - \left[\frac{1}{x-2)}\right] = \frac{1}{(x-2)^2}$$

therefore: $\int_3^7 \frac{dx}{(x-2)} = \left[\frac{-1}{x-2}\right]_3^7 = \frac{8-}{10}$

(iii) $\int_0^1 (x^{3/4} + ½ \text{ x } ½)dx$

let F(x) = $4/7x^{3/4}$ + ½ x ½) = 4/7 x 7/4 + 1/3 x 3/2

therefore $\int_0^1 (x^{3/4} + ½ \text{ x } ½) = [\, 4/7\ x^{7/4} + 1/3\ x^{3/4})_0^1 = \underline{19}$

(iv) $\int_0^9 (a^2x^2 - x^4)dx$

let F(x) = $\left(\frac{a^2x^3}{3} - \frac{x^5}{5}\right)$

$\int^a (a^2x^2 - x^4)dx = \left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]_0^a$

$= \frac{a^3x^3}{3} - \frac{a^5}{5}$

(v) $\int_1^3 \frac{2-x}{x^3} dx = \int^3 \left(\frac{2}{} - \frac{1}{}\right)dx$

Let F(x) = $\frac{1}{x} - \frac{1}{x^2}$

then $\int_1^3 \frac{2-x}{x^3} = \left[\frac{1}{x} - \frac{1}{x^2}\right]_1^3 = \frac{9}{2}$

(vi) $\int^8 (\surd E - 1/t^2)dt$

Let F(t) = $\frac{2t^{3/2}}{3} + \frac{1}{t}$

$\int_1^8 \surd t - 1/t^2 = \left[\frac{2t^{3/2}}{3} + \frac{1}{t}\right]_1^8 = \frac{32\surd 2}{3} - \frac{37}{24}$

(vii) $\int_1^3 \left(\frac{6-t}{t^4}\right)dt = \int_1^3 \left(\frac{6}{t^4} - \frac{1}{t^3}\right)dt$

F(t) = $\underline{1} - \underline{2}$

$\int_1^3 \left(\frac{6}{t^4} - \frac{1}{t^3}\right)dt = \left[\frac{1}{2t^2} - \frac{2}{t^3}\right]_1^3 = \frac{40}{27}$

(x) $\int_1^2 (x - 1)\, x^2\, dx = \int_1^2 (x^3 - x^2)dx$

$F(x) = \frac{x^4}{4} - \frac{x^3}{3}$

$\int_1^2 (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = \frac{17}{12}$

(xi) $\int_1^4 \sqrt{x} + 1dx$

$F(x) = 2/3(x + 1)^{3/2}$

then $\int_1^4 \sqrt{x} + 1 = [2/3(x + 1)^{3/2}]_1^4 = \frac{10\sqrt{5}}{3} - \frac{4\sqrt{2}}{3}$

(xii) $\int^1 (x - 1)^7 dx$

$F(x)\ \frac{1}{8}(x - 1)^8$

therefore: $\int_0^1 (x - 1)^7 = \left[\frac{1}{8}(x - 1)^8\right]_0^1 = \frac{-1}{8}$

4.0 CONCLUSION: In this unit, you have studied how to define a definite integral. You have seen the connection between the summation process of finding the area under a curve and the differentiation of the function representing the area under the curve. You have studied that the fundamental theorem of integral calculus is the bridge between the summation process and the differentiation process i.e. you can find the area under a curve by finding an anti-derivative for the curve. You have applied the theorem in evaluation of definite integrals.

5.0 SUMMARY: You have studied the following in this unit:

1. How to define a definite integral
2. How to evaluate definite, integral using the following properties:

   (i) $\int_a^a f = 0$, (ii) $\int_a^b f + \int_b^c = \int_a^c$ and (iii) $\int_a^b f = -\int_b^a f$

3. How to apply the fundamental theorem of integral calculus in evaluating the definite integral of rational functions.

6.0 TUTOR MARKED ASSIGNMENTS

Evaluate the following integrals by applying the fundamental theorem of integral calculus.

(1) $\int_0^1 (4x - 3)dx$

(2) $\int_1^0 5x - 3dx$

(3) $\int_0^1 (3x + 2)dx$

(4) $f^5\sqrt{x}$

(5) $\int_{-c}^{a} (x - a)^2 dx$

(6) $\int_1^2 (\underline{5} + t^x)dx$

(7) $\int_0^2 (1 - x)dx$

(8) $\int_{-4}^{-t} (\frac{1}{x^2} + x)\, dx$

(9) $\int_{-2}^{-1} \frac{1}{x^4}$

(10) $\int_{-2}^{2} (3 + 2x - x^2)dx$

(11) $\int_1^2 (\sqrt{x} - \frac{1}{\sqrt{x}})dx$

(12) $\int_1^2 (3t + 4t^2)dt$

(13) $\int_1^3 (\frac{x^2 + 1}{x^2})dx$

(14) $\int_0^1 x^2(x - 1)dx$

(15) $\int_1^4 (t^3 - t)dt$

(16) $\int_{-2}^{1} (x + 1)\,(x - 2)dx$

(17) $\int_1^2 x^{-1/2}dx$

(18) $\int_1^2 \frac{2(x + 3)}{x^3}dx$

(19) $\int_1^3 (\sqrt{x} + \frac{1}{\sqrt{x}})^2$

(20) $\int_2^3 (2v - 3\sqrt{v})dv$

UNIT 3: INDEFINITE INTEGRAL

1.0 INTRODUCTION

You have studied rules for differentiation of various functions such as polynomids functions, rational functions, trigonometric function of sines, cosines, tangent etc. hyperbolic functions and then inverses, exponent and logarithm functions. All these you studied in the first course in calculus. However, the reverse process i.e. anti-differentiation is some how not as straight forward process as the differentiation. The reasons being that there are no systematic rules or procedures for anti-differentiation. Rather success on techniques of anti-differentiation depends much more on your familiarity with differentiation itself. So before embarking on the study of this unit, it might be worth the time to practice some of the differentiation in calculus I. Do not be discouraged when you come across functions whose derivatives are not very common. In this unit and subsequent ones you will study some basic methods of anti-differentiation.

2.0 OBJECTIVES

After studying this unit, you should be able to:

(i) evaluate indefinite integral as anti-differentiation.
(ii) Recall notations for integration and
(iii) Recall properties of indefinite integration
(iv) Evaluate indefinite integrals using the properties of indefinite integration.
(v) Integrate differential equations that are separable.

3.0 INDEFINITE INTEGRATION

In this section an informal definition of what is anti-differentiation will be given. Suppose that the derivative of the function is given as:

$$\frac{dy}{dx} = f(x)$$

and you were asked to find the function $y = F(x)$. For example you are given the differential equation. $\frac{dy}{} = 2x$.

dx

From your experience with differentiation you can easily know that $y = x^2$ since $\frac{dy}{dx} = 2x$

Interestingly, it is not only $y = x^2$ that can be differentiated to give $\frac{dy}{dx} = 2x$.

Other function like $y = x^2 - 1$, $y = x^2 + 2$, $y = x^2 + a$, $y = x^2 + 4$ can be differentiated to yield $\frac{dy}{dx} = 2x$

In general any function of this form $y = x^2 + c$, where C is any constant will yield a differential equation of this type $\frac{dy}{dx} = 2x$

You are now ready to take this definition.

Definition 1: An equation such as $\frac{dy}{dx} = f(x)$ which specifies the derivative as a function of x (or as a function of x and y) is called a differential equation. For example $\frac{dy}{dx} = \text{sine}$ is differential equation

Definition 2: A function $y = F(x)$ is called a solution of the differential equation $\frac{dy}{dx} = f(x)$ if over domain a<xcb F(x) is differentiable and

$\frac{d}{dx} F(x) = F^1(x) = f(x)$

in this case F(x) is called an integral of f(x) with respect to x.

Definition 3: If F(x) is an integral of the function f(x) with respect to x so is the function F(x) + C an integral of f(x) with respect to x, where c is an arbitrary constant. If $\frac{d}{dx} F(x) = f(x)$ so also is F(x) + C i.e. $\frac{d}{dx}[F(x) + C] =$

$\frac{d}{dx} F(x) + C \frac{dc}{dx} = \frac{df(x)}{dx} + 0 = F^1(x) = f(x)$

From the above if $y = F(x)$ is any solution of $\frac{dy}{dx} = f(x)$ then all other solutions are contained in the formula $y= F(x)+C$ where C is an arbitrary constant this gives rise to the symbol. $\int f(x)\,dx = F(x) + C$ – (1) where the symbol *S* is called an integral sign (see unit 2). Equation 1 is read the integral of f(x)dx is equal to F(x) plus C since $\frac{dy}{dx} = 2x$ and a typical

solution is $F(x) = x^2 + C$. then $\frac{d\,F(x)}{dx} = 2x = \frac{d(x^2+C)}{dx}$

$= y = x^2 + C$

and $\frac{dy}{dx} = \frac{d}{dx}(x^2 + C) = 2x$

Example: If $y = x$ $\frac{dy}{dx} = 1$

$= \int \frac{dy}{dx} dx = \int 1 dx = x + C$

In other words, when you integrate the differential of a function you get that function plus an arbitrary constant.

Example: Solve the differential equation $\frac{dy}{dx} = 4x^3$

Solution: let $\frac{dy}{dx} = 4x^3$

then $dy = 4x^3dx$ integrate both side you get $\int dy = \int 4x^3dx$ but $\frac{d}{dx}(x^2) = 4x^3dx$.

therefore $y = \int 4x^3dx = \int d(x^4) = x^4$ $_{C.}$

Example: Solve the differential equation $\frac{dy}{dx} = 2x + 1$

$= dy = (2x + 1) dx$

but $\frac{d}{dx}(x^2 + x) = 2x + 1$

therefore $\int dy = \int (2x + 1)dx$ becomes $y = \int dx^2 + x) = x^2 + x + C$.

compare $\int d(F(x)) = F(x)$ with the result of UNIT 2.

Example: Solve the following differential equation:

(1) $\frac{dy}{dx} = x^2 - 1$ (2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

(3) $\frac{dy}{dx} = \frac{x}{y}$ (4) $\frac{dy}{dx} = 2x + 3$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})dx$ (6) $\frac{dy}{dx} = 3x^2-2x+3$

(7) $\frac{ds}{dt} = 3t^2-2t-6$ (8) $\frac{dv}{du} = 5u^4-3xu^2-1$

(9) $\underline{dx} = 8\sqrt{x}$ (10) $\underline{dy} = (2x^2- \underline{1})$

dt dx x^2

Solution: $\frac{dy}{dx} = x^2 - 1$

$= dy\ (x^2 – 1)dx$

$\int dy = S(x^2-1)dx$

but $d(\frac{x^3}{3} – x) = (x^2-1)dx$

therefore: $y = \int d(\frac{x^3}{3}-x) = \frac{x^3}{3} – x + C$

(2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

$\int dy = S(\frac{1}{x^2} + x)dx$

$d(\frac{-1}{x} + \frac{x^2}{2}) = (\frac{1}{x^2} + x)\ dx$

$y = \int d(\frac{x^2}{2} – \frac{1}{x}) = \frac{x^2}{2} - \frac{1}{x} + c$

(3) $\frac{dy}{dx} = \frac{x}{y}$

$\int ydy = \int x\ dx$

$d(\frac{y^2}{2}) = ydy$ and $d(\frac{x^2}{2}) = xdx$

therefore: $\int ydy = \int d(\frac{y2}{2})$

$\int d(\frac{y^2}{2}) = \int d(\frac{x^2}{2})$

$= \frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 = x^2 + 2C_1$

$y^2 = x^2 + C$

(4) $\frac{dy}{dx} = 2x + 3$

$\int dy = \int (2 + 3)\, dx$

$y = \int d(x^2 + 3x) = x^2 + 3x + C$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})$

$dy = (x^2 + \sqrt{x})\, dx$
$\int dy = \int (x^2 + \sqrt{x})dx$
$y = \int d(\frac{x^3}{3} + \frac{2}{3}x^{3/2}) = \frac{x^3}{3} + \frac{2}{3}x^{3/2} + C$

(6) $\frac{dy}{dx} = 3x^2 - 2x - 5$

$\underline{dy} = 3x^2 - 2x - 5$

$\int dy = \int (3x^3 - x^2 - 5x) = x^3 - x^2 - 5x + C$

(7) $\frac{ds}{dt} = 3t^2 - 2t - 6$

$\int ds = \int (3t^2 - 2t - 6)dt$

$\int = \int d(t^3 - t^{2-}6t) = t^3 - t^2 - 6t + C$

(8) $\frac{dv}{du} = 5u^4 - 3u^2 - 1$

$\int dv = \int (5u^4 - 3u^2 - 1)du$
$V = \int d(u^5 - U^{3\,-}U) = U^5 - U^3 - U + C$

(9) $\frac{dx}{dt} = 8\sqrt{x}\, dt$

$dx = 8\sqrt{x}\, dt$

$= \int \frac{dx}{\sqrt{x}} = \int 8\, dt$

$\int d(2\sqrt{x}) = \int d(8t)$
$2\sqrt{x} + C_x = 8t + C_t$
$2\sqrt{x} = 8t + C_x + C_t$

$2\sqrt{x} = 8t + C$, where $C = C_x + C_t$

(10) $\frac{dy}{dx} = (4x^2 - \frac{1}{x^2})$

$\int dy = \int (4x^2 - \frac{1}{x})dx$

$y = \int d(\frac{4x^3}{3} - \frac{1}{x}) = \frac{4x^3}{3} - \frac{1}{x} + C$

Exercise: Evaluate the following:

(1) $\int \frac{dx}{x^5}$

(2) $\int (x + 1)^3 dx$

(3) $\int (ax^2 + b)dx$

(4) $\int (\frac{x^3 + 1}{x^6})\, dx$

(5) $\int (\frac{x^3 - 1}{x^2})dx$

(6) $\int (t^2 - a)(t^2 - b)dt$

(7) $\int (\sqrt{x} - \frac{1}{x^{1/3}})dx$

(8) $\int (\frac{5x}{x^5})^4 dx$

(9) $\int \frac{dx}{\sqrt{1+x}}$

(10) $\int (x - 1)^2 + \frac{1}{(x+2)^2})dx$

Ans:

(1) $\frac{-1}{4x^4} + C$

(2) $\frac{1}{4}(x + 3)^4 + C$

(3) $1/3ax^3 + bx + C$

(4) $\frac{-1}{10}\frac{5x^3 + 2}{x^5} + C$

(5) $\frac{1}{2}\frac{(x^3 + 2)}{x}$

(6) $\frac{1}{5}t^5 + \frac{1}{3}(b-a)t^3 - abt + C$

(7) $2/3\, x^{3/2} - 3/2\, x^{2/3} + C$

(8) $\frac{125}{3x^{15}} + C$

(9) $2\sqrt{x} + 1 + C$

(10) $1/3(x - 1)^3 - \frac{1}{(x+1)} + C$

7.1 Properties of Indefinite Integral

So far, you would have been doing much of guess work to find an appropriate anti-derivative that will fit the answers above you will now be given some properties of indefinite integral. It would help reduce the amount of guesswork when evaluating integrals.

(1) The integral of the differential of a function U is U plus an arbitrary constant. $\int du = u + c$

(2) A constant may be moved across the integral sign $S a du_1 = a S du$

(3) The integral of the sum of two differentials is the sum of their integrals $\int (du + dv) = \int du + \int dv$

(4) The integral of difference of two differential is the difference of their integrals $\int (du - dv) = \int du - \int dv$

(5) As a consequent of 2, 3 and 4 above, you have that
$\int a(du \pm dv) = a\int du \pm a \int dv$

$\int du_1 \pm du_2 \pm du_3 \ldots du_h = \int du_1 \pm \int du_2 \pm .. \pm \int du_h$

(7) If n is not equal to minus 1, the integral of $U^n du$ is obtained by adding one to the exponent dwindling by the new exponent and adding an arbitrary constant $\int u^n du = \frac{U^{n+1}}{n+1} = C$

Find the following

Example (1) $\int (5x^{10} - x^8 + 2x)dx = \int 5x^{10}dx - \int x^8 dxt \int 2xdx$

$$= \frac{5x^{10+1}}{10+1} - \frac{x^{8+1}}{8+1} + \frac{2x^{1+1}}{1+1}$$

$$= \frac{5}{11}x^{11} - \frac{x^9}{9} + x^2 + C$$

(2) $\int x^{3/2}dx = \frac{x^{3/2+1}}{^3/_2{}^{+1}} = \frac{x^{5/2}}{^5/_2}$

$$= 2/5\ x^{5/2}$$

(3) $\int \quad 3x + 1dx$
Let $u = 3x+1$

then $\frac{du}{dx} = 3 = du = 3dx$

Therefore $\int\sqrt{3x+1}dx = \int u^{1/2} \frac{du}{3}$

here $dx = \frac{du}{3}$

therefore $1/3 S u^{1/2} du = 1/3 \frac{u^{1/2+1}}{½+1}$

$= \frac{1}{3} \frac{2u^{3/2}}{3} = \frac{2U^{3/2}}{9}$

$= \frac{2}{9}(3x+1)$

(4) $\int\sqrt{-} 1 dx$

Let $U = 4x – 1$ $\frac{du}{dx} = 4 = dx = \frac{du}{4}$

then $\int\sqrt{4x-1}dx = \int u^{1/2} \frac{du}{4}$

$¼ \int u^{1/2} du = \frac{U^{1/2+1}}{½+1} = \frac{2}{12}(4x-1)^{3/2} + C$

Examples: Evaluate the following integrals

(i) $\int\sqrt{1=4x}$ (ii) $\int 3\sqrt{1 + x}\, dx$

(iii) $\int \sqrt[5]{2x+1}$ (iv) $\int \sqrt[4]{4x - 2}$

(v) $\int \sqrt[6]{6x + 4}$

Solution:

(i) $\int \sqrt{1-4x}dx$ let $U = 1- 4x$

then $\frac{du}{dx} = 4$, $dx = \frac{-du}{4}$

therefore $\int\sqrt{1-4x}\, dx = \int u^{1/2}\frac{(-du)}{} = - ¼ \int u^{1/2}du$

$= - \frac{1}{4}\left[\frac{2u^{3/2}}{3}\right] = - \frac{1}{6}(1-4x)^{3/2} + C$

(ii) $\int \sqrt[3]{1+x}\,dx$

then $\frac{du}{dx} = 1$ du = dx

therefore $\int \sqrt[3]{1+x}\,dx = \int u^{1/3}du = \frac{3}{4}u^{4/3} = \frac{3}{4}(1+x)^{4/3} + C$

(iii) $\int \sqrt[5]{2x\ 1}\,dx$ let U = 2x + 1

then du/dx = 2 $\Rightarrow$ = dx = du/2

therefore $\int \sqrt[5]{2x+1}\,dx = \int U^{1/5}\frac{du}{2} = ½\left[\frac{5U^{6/5}}{6}\right] = \frac{5}{}(2x+1)^{6/5}C$

$= \frac{5}{12}(2x+1)^{6/5} + C$

(iv) $\int \sqrt[4]{4x-2}\,dx$ let U = 4x - 2

then $\underline{du}$ = 4dx $\Rightarrow$ dx = du/4

$\int\ 4\ \ 4x\ -2dx = ¼ \int U^{1/4}\,du = ¼\ [4/5(4x-2)^{5/4}]$

$= 1/5\ (4x-2)^{5/4}$

(v) $\int \sqrt[6]{6x+4}\,dx$ let U = 6x + 4

then dx = du/6

$\int \sqrt[6]{6x+4}\,dx = 1/6 \int U^{1/6}\,du = 1/6.6/7\ (6x+4)^{7/6}+C$

$= 1/7\ (6x+4)^{7/6} + C$

Exercise: Evaluate the integrals

(1) $\int (8x^7 - 6x^5 - x^4 + 3x^3 + 2)\,dx$

(2) $\int (6x+1)^{1/6}dx$

(3) $\int (1-4x)^{1/4}dx$

(4) $\int (4-10x)^{1/10}dx$

(5) $\int (x-1)^{1/3}dx$

Ans:

(1) $x^8 - x^6 - \frac{x^5}{5} + \frac{3x^4}{4} + 2x + C$

(2) $\frac{1}{7}(6x+1)^{7/6} + C$

(3) $\frac{-1}{5}(1 - 4x)^{5/4} + C$

(4) $\frac{-1}{11}(4 - 10x)^{11/10} + C$

(5) $\frac{3}{4}(x - 1)^{4/3} + C$

3.2 Application of Indefinite Integration

Most elementary differential equation could be solved by integrating them.

Example: Solve the differential equation given as $\frac{dy}{dx} = f(x)$

$\Rightarrow$ $dy = f(x)\,dx$

$\Rightarrow$ $\int dy = \int f(x)\,dx$

$\Rightarrow$ $y = \int f(x)\,dx$

Such class of differential equation is used to solve various types of problems arising from Biology, all branches of engineering, physics, chemistry and economics.

In application of indefinite integral the value of the arbitrary constant must be found by applying the initial conditions of the problem that is being solved. Therefore before continuing it is important that you know more about this arbitrary constant C.

Example: Let $\frac{dy}{dx} = 2x$

then $y = x^2 + C$

The graph of $y = x^2$ for $C = 0$ is given in Fig. 3.1

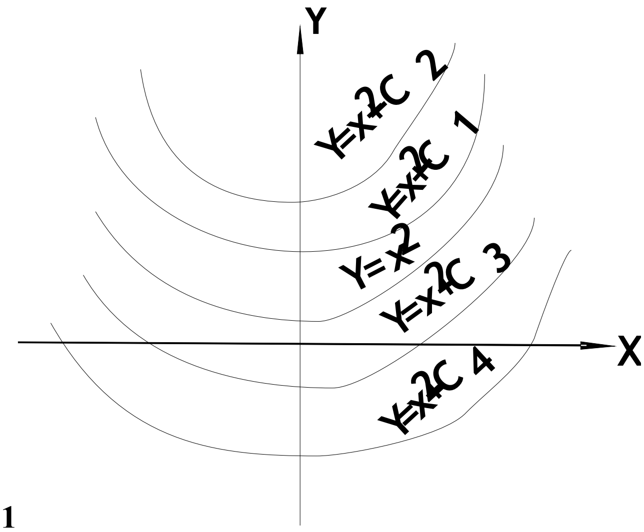


Fig. 3.1

Any other integral curve $y^2 + C$ can be obtained by shifting this curve $y = x^2$ through a vertical displacement C. In Fig.3.1 such vertical displacements give rise to a family of parallel curves. They are parallel since the slope of each curve is equal to 2x. This family of curves has the property that for any given part (x_o, y_o) where $x_o \in {}^1D$ (i.e. D is the domain of definition) there is only one and only one curve from the family of curves that passes through the part (x_o, y_o). Hence the part (x_o, y_o) must satisfy the equation

$$Y_o = x_o^2 + C$$

i.e. $C = y_o - x_o^2$ so for any particular point $(x_o\ y_o)$ C can be uniquely be determined.

This condition that $y = y_o$ and $x = x_o$ imposed on the differential equation du/dx = 2x is referred to as initial condition. You will use this method to solve problems on application of integration.

Example: Total profit P(x) from selling X units of a product can be determined by integrating the differential equation of the marginal profit dp/dt and using some initial conditions based on the market forces to obtain the constant of integration. Given that

$$dp/dt = 2 + 3/(2x-1)^3$$

Find P(x) for $0 \le x \le C$ if P(1) = 1.

Solution:

$$dp = \int \left(2 + \frac{2}{(2x-1)^3}\right)dx = \int \left(2 + \frac{2}{(2x-1)^3}\right)dx$$

$$\int dp = \int \left(2 + \frac{2}{(2x-1)^3}\right) dx$$

$$P = 2x - \frac{-1}{2(2x-1)^2} + C$$

Since $P_o = 1$ $X_o = 1$

$$1 = 2.1 - \frac{1}{2(2-1)^2} + C$$

$\Longrightarrow C = -1/2$

Therefore $P(x) = 2x - \frac{1}{2(2x-1)} - 1/2$

Example: Given that $dy/dx = 8x^7$

Find y when y = -1 and x = 1

Solution:

$$\int dy = \int 8x^7\, dx$$

$$y = x^8 + C$$

$$-1 = 1 + C$$

$$C = x^8 - 2$$

Exercises:

Solve the following equations subject to the prescribed initial conditions:

(1) $dy/dx = 4x^2 - 2x - 5$ $\quad x = -1, y = 0$

(2) $dy/dx = 4(x-5)^3$ $\quad x = 0, y = 2$

(3) $dy/dx = \frac{x^2+1}{x^4} dx$ $\quad x = 1, y = 1$

(4) $dy/dx = x\sqrt{1+x^2}$ $x = 0, y = 0$

(5) $dy/dx = x^{1/2} + x^{1/5}$ $x = 0, y = 2$

You will study more on application of indefinite integration in the last unit in the course.

Ans: (1) $y = 4/3x^3-x^2+5x+22/3$ (ii) $y = (x-5)^4-623$

(iii) $y = -1/x - 1/3x^3+7/3$ (iv) $y = 1/3(x^2+1)^{3/2}-1/3$

(v) $y = 2/3x^{3/2}$

4.0 In this unit emphasis has been on techniques of finding anti-derivative. Therefore, you have studied numerous solved examples on method of finding anti-derivatives of functions. You have known the notation for indefinite integration as $\int f(x)dx = f(x) + C$. You have studied properties of indefinite integration and how to use them to evaluate integrals. You have studied how to integrate simple differential equations.

5.0 Summary

You have studied:

(1) the definition of indefinite integral
(2) Properties and notation of indefinite integration
(3) To evaluate integrals using both the notation and properties of indefinite integration.
(4) To integrate differential equation that are separable.

6.0 Tutor Marked Assignments

Evaluate the following integrals:

(1) $\int \sqrt{}dx$ (2) $\int \sqrt{4x\ 1}$

(3) $\int (7x^6-4x^3 + 4x^6 - 2x)$ (4) $\int dx/x^7\, dx$

(5) $\int \frac{x^4 - 1}{X^6} dx$ (6) $\int \left(\sqrt{x} + \frac{1}{\sqrt{1+x}}\right) dx$

(7) $\int \frac{(5x - 1)^2}{x^3} dx$ (8) $\int \frac{4x^3 - 1}{x^6} dx$

$\Big(\qquad \Big)$

(9) $\int (\sqrt{4x+1} - \sqrt{3x})\, dx$ (10) $\int x^2 + \frac{x^2 + 2x}{x2}\, dx$

(11) $\int (1 - 8x)^{1/8} dx$ (12) $\int (5x - 2)^{1/5} dx$

Solved the differential equation at the specified points:

(13) $dy/dx = \frac{x^2 - 1}{x^4}$ $y = 0\ x = 1$

(14) $dy/dx = \frac{1}{\sqrt{1+7x}}$ $y = 2,\ x = 1$

(15) $dy/dx = (1 - 4x)^{1/4}$ $y = 1, x = -3$

(16) $dy/dx = 6c\sqrt{1 - x^2})dx$ $y = 0, x = 1$

(17) Find the total profit of a product if the marginal profit is given as $dp/dx = x^4 + x^2 (\sqrt{1-x^3})$
where $P(0) = 0$

(22) Solve $dy/dx = \frac{2\sqrt{1+y^2}}{y}$ if $x = 1, y = 1$

(19) Solve $dy/dx = x^2/y^3$ if $x = 0, y = 1$

(20) Solve $ds/dt = (t^2+1)^2$ when $S = 0, t = 0$

UNIT 4

INTEGRATION OF TRANSCEDENTAL FUNCTIONS

1.0 INTRODUCTION

2.0 OBJECTIVES

3.0 INTEGRATION OF RATIONAL AND EXPERIMENTAL

3.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

3.2 INTEGRATION BY INVERSE TRIGONOMETRIC FUNCTIONS

4.0 CONCLUSION

5.0 SUMMARY

1.0 INTRODUCTION:

In the previous unit, you studied the integration of polynomial function and simple rational function. However, there are some functions whose derivatives are not very common. Integration of such functions uncommon derivatives can only be possible by using derivatives of known functions to do the evaluation. In this unit integration of transcendental and rational function are discussed. These integration will form part of the basic tools that will be needed in applying techniques of integration that will be studied in the next unit.

2.0 OBJECTIVES

After studying this unit you should be able to correctly

(i) derive the formula for integrating rational functions, exponential function and trigonometric functions

(ii) evaluate definite and indefinite integrals of sinx, cos x, e^x and any combination of them

(iii) to evaluate integrals by using the derivatives of inverse trigonometric functions of sinx and tan x.

3.0 INTEGRATION OF RATIONAL AND EXPONENTAL FUNCTION

3.0.1 The integral $\int du/u =$, $u \neq 0$ Recall that d/dx 1nu = du/u - I

(see unit 8 of calculus I)

then the integral counterpart of equation I above is that $\int du/u = \ln 1u1 + C$

In the above u is a differentiable function of x and u>0 for all values of x in the specified domain.

Example: Find $\int \frac{8x}{2^{x-1}} dx$

Solution: let $u = x^{2-1}$, $du = 2x\, dx$

then

$$\frac{du}{2} = x\, dx$$

therefore

$$\int \frac{8x\, dx}{x^{2-1}} = \frac{8 \int du}{2}$$

$$= 4\int \frac{du}{x} = 4/u/u\int + C$$

Example: Find $\int \frac{x^2}{1+3x^3} - dx$

let $u = 1+3x^3$, $du = 9x^2dx$

$\Rightarrow x^2dx = du/9$

$$\int \frac{x^2dx}{1_{3x}^{3}} = \int \frac{\frac{du}{9}}{u} = \frac{1}{9}\int \frac{du}{u}$$

$$= \frac{1}{9}\ln/u/ + C = \frac{1}{9}\ln/1+3x^3/+C$$

Example: Find

$$\int \frac{8x^3 - 2}{x^4-x+1} dx$$

let $u = x^4 - x + 1$, $du = (4x^3 - 1)\, dv$

but $(8x^3 - 2)dx = 2(4x^3-1)\, dx$

therefore: $\int \frac{(8x^3 - 2)dx}{x^4-x+1} = \int \frac{2(4x^3-1)dx}{x^4-x+1} = \int \frac{2du}{u}$

$= 2\ln/u/+C$

$= 2\ln/x^4 - x + 1/ + C$

Example: Find $\int \left(\frac{1}{(x+1} - \frac{1}{x+2)}\right) dx$

let $u = x + 1$ and $v = x + 2$
$du = dx$ $du = dx$

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \int \frac{du}{u} - \int \frac{dv}{v}$$

$$= \ln/u/ - \ln/v/ + C$$

$$= \ln/x+1/ - \ln/x+2/ + C$$

Example: Find $\int \frac{\log(x+1)}{x+1} dx$

let $u = \log(x+1)$ $du = \frac{1}{x+1} dx$

therefore: $(x+1)\, du = dx \Longrightarrow \int \frac{\log(x+1)}{x+1} dx = \int \frac{u}{x+1} .(x+1) du$

$$= \int u du = \tfrac{1}{2} u^2 + C$$

$$= \tfrac{1}{2} \log (x+1)^2 + C$$

Exercise: Evaluate the following integrals

(1) $\int \frac{dx}{3-4x}$ (2) $\int \frac{3}{x-5} dx$

(3) $\int \frac{x}{x^2-2} dx$ (4) $\int \frac{\log x}{x} du$

(5) $\int \frac{4x-2}{x^2-x+1} dx$

Ans: (1) $-\frac{1}{4} \ln/(3-4x)/+C$ (2) $3\ln/(x-5) + C$

(3) $\tfrac{1}{2} \ln/x^2-2/ + C$ (4) $\tfrac{1}{2} \log x^2 + C$

(5) $2\ln/x^2 - x + 1/$

The method adopted above is to differentiate the denominator and check if it is a factor of the numerator; if so with appropriate algebraic manipulation, the derivative of the denominator will be made to look like the numerator. This method was used in UNIT 3.

i.e. $\int \frac{g(x)}{P(x)} dx$ let $u = P(x)$

and $du = P^1(x)dx = g(x)dx$

then $\int \frac{g(x)}{P(v)} = \int \frac{du}{u} = \ln/u/ + C$

$\Longrightarrow$ $Ln/P(x) + C$

3.0.2 THE INTEGRAL $\int e^x du$

Recall that $\frac{de^u}{dx} = \frac{de^u}{du} \cdot \frac{du}{dx} = e^u \frac{du}{dx}$

then $\frac{de^u}{dx} = e^u \frac{du}{dx}$

$\Longrightarrow$ $de^{u\,=}\, e^u du$

then $\int de^u = \int e^u du$

therefore $\int e^u du = e^u + C$

Example: Find $Se^{-x}dx$.

Let $u = -x$, $du = 1dx$

$\Longrightarrow$ dx -du

therefore $\int e^{-x}dx = \int e(-du) = -\int e^u du$

$= -e^u + C = e^{-x}$ C.

Example: Find $\int e^{2x}dx$. Let $u = 2x$ $\Longrightarrow$ $du = 2dx$

$dx = \frac{du}{2}$

therefore $\int e^{2x}dx = \int e^u(\frac{du}{2}) = \frac{1}{2} \int e^u du$

$= \frac{1}{2} e^{2x} + C$

Example: Find $\int e^{x/3}dx$ let $u = \underline{x}$, $du = \underline{dx}$

$$dx = 3du, \; S e^{x/3} \; \int e^{x/3} du = \int e^{u} (3du)$$

$$\int e^{x/3} dx = 3 S e^{u} du = 3e^{u} \; C$$
$$= 3e^{x/3} + C$$

Example: $\int 4e^{2x} dx$ Let $U = e^{2x}$ $du = 2e^{2x} dx$.
$\int 4e^{2x} du = 2\int 2e^{2x} dx = 2\int du = 2u + C$
$= 2e^{2x} + C$

Example: $\int (e^{x} \; x)^{2} (e^{x} \; 1) dx$
Let $u = e^{x} + x = du = (e^{x} \; _{1})dx$
$\int (e^{x} + x)^{2} (e^{x} + 1) dx = \int u^{2} du$
$= \frac{U^{3}}{3} + C = \frac{(e^{x} + x)^{3}}{3} + C$

Example: $\int xe^{x2} dx$
Let $u = x^{2}$ $du = 2x \; dx$
$\Rightarrow \frac{du}{2} = x \; dx$
then $\int xe^{x2} dx = ½ S e^{u} du$
$= ½ e^{u} + C = ½ e^{x2} + C$

Exercise: Evaluate the following integrals

(1) $\int e^{3x} dx$ (2) $\int e^{\frac{sx}{2}}$

(3) $\int 8e^{4x} dx$ (4) $\int (e^{x} – x)^{2} (e^{x} – 1) dx$

(5) $\int 3x^{2} e^{x3}$

Ans: (1) $1/3e^{3x} + C$ (2) $\frac{2}{5} e^{\frac{5x}{2}} + C$ (3) $2e^{4x} + C$

(5) $e^{x3} + C$

3.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Recall from UNIT 8 of the first course on calculus that for any differentiable function U of X that

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = \sin^2 U\ \frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = -\mathrm{cosec}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\mathrm{cosec}^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u\ \frac{du}{dx}$$

$$\frac{d}{dx}(a\sec u) = -w\sec u\cot u\frac{du}{dx}$$

Using the above you will integrate the following trigonometric function as

(1) $\int \sin u du = -\int -\sin u du = -\int \frac{d}{dx}(\cos u)$

$= -\cos u\ \ C$

therefore $\boxed{\int \sin u\ du = -\ \mathrm{Cos} u + C}$

(ii) $\int \mathrm{Cos} u\ du = \int d(\sin u) = \sin u\ \ C$

therefore $\boxed{\int \cos u du = \sin u + C}$

Given that $\int \frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)]\ dx = \log/f(x) + C$

then

(iii) $\int \tan u du = \int \frac{\sin u}{\cos u} du = -\int \frac{1}{\cos u}.d \cos u)$

$= -\int \frac{dv}{v} = \ln v + C$, where $v = \cos u$

$= -\ln k o s u l + C = \ln(\frac{1}{\cos u}) = \ln/\sec U/+C$

therefore $\int \tan u\ du = \ln/\sec u/ + C$

(iv) $\int \sec u du = \int \sec u\frac{(\sec u + \tan u)}{(\sec u + \tan u)}du$

$= \int \frac{\sec^2 + \sec u \tan u}{\sec u + \tan u} du$

Let V = tan u + sec u, dv = $\sec^{2u}$ + tan u sec u du

therefore: $\int \frac{\sec^{2u} + \tan u \sec u\, du}{\tan u + \tan u} = \int \frac{dv}{v}$

(v) $S\cot u du = \int \frac{\cos u}{\sin u} du = \int \frac{1}{\sin u} d(\sin u)$

= ln/sinu/ + C

(vi) $\int \operatorname{cosec} u\, du = \int \operatorname{cosec} u \frac{\operatorname{cosec} u - \cot u}{\operatorname{cosec} U - \cot U}$

$= \int \frac{\operatorname{cosec}^{2u} - \cot u \operatorname{cosec} u}{\operatorname{cosec} u - \cot u} du$

$= \int \frac{dv}{v}$, v = cosec u – cot u

dv = $\operatorname{cosec}^{2u}$ – cot cosec u du

$\Rightarrow$ ln/v/ + C = ln/cosec u – cot u/ + C.

Example: Find $\int \sec^2 u\, du = \int d(\tan u)$ C

= tan u + C

Example: Find $\int \operatorname{cosec}^2 u\, du = -\int -\operatorname{cosec}^2 u\, du$ $-\int$ (d(cot u) = -cotu + C

Example: Find $\int \sec u \tan u\, du$ $\int \sec u \tan u = \int d(\sec u) = \sec u + C$

Example: Find $\int \cos x \sin x\, dx$

Let u = sinx du = cos x dx

therefore $\int \sin x \cos x\, dx = \int u\, du$

$= \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$

Example: Find $\int \sec^3 x \tan x\, dx$

Let u = sec x du = sec x tan x dx

therefore $\int \sec^3 x \tan x\, dx = \int \sec^2 x \sec x \tan x\, dx$

$= \int u^2 du = \frac{u^3}{3} + C$

$= \frac{\sec^3 x}{3} + C$

Example: Find $\int \operatorname{cosec}^3 x \cot x\, dx$. Let u = cosec x du = -cosec x cot x dx

Therefore $\int \operatorname{cosec}^3 x \cot x\, dx = \int \operatorname{cosec}^2 x \operatorname{cosec} x \cot x\, dx$

$= -\int u^2 du = \frac{-u^3}{3} + C$

$= \frac{-cosec^3 x}{3} + C$

Example: Find $\int x\cos ax^2\, dx$

Let $U = ax^2$ $du = 2axdx$

$\int x\cos ax^2 dx = \int ½ a (\cos ax^2)(2ax)\, dx$

$= ½ a \int \cos U.du = ½ a (\sin U + C)$

$= ½ a \sin ax^2 + C$

Example: Find $\int \frac{\sec^2 x}{1+\tan x} dx$

let $U = 1 + \tan x$ $du = \sec^2 x\, dx$

therefore $\int \frac{\sec^2 x dx}{1+\tan x} = \int \frac{du}{u} = \ln/u/+C$

$= \ln/1+ \tan x1 + C$

Exercises: Find the following integrals

(i) $\int \sin(2x-1)\, dx$ (ii) $\int \sin ½ ax\, dx$

(iii) $\int 2\cos^2 x \sin x\, dx$ (iv) $\int \sin^4 x \cos x\, dx$

(v) $\int x\tan x^2\, dx$ (vi) $\int \frac{dx}{\cos^{2x}}$

(vii) $\int \frac{\sin x}{1 + \cos x} dx$ (viii) $\int \cot ax\, dx$

(ix) $\int \cos^6 ax \sin ax\, dx$ (x) $\int (1+\tan x) \sec^2 x\, dx$

Ans: (i) $- ½ \cos (2x – 1) + C$

(ii) $\frac{-2}{a} \cos ½ ax + C$

(iii) $\frac{-2}{3}\cos^3 x + C$

(iv) $\frac{1}{5}\sin^5 x + C$

(v) $½ \ln/\sec x^2/+C.$

(vi) $\tan x + C$

(vii) $-\ln(\cos x + 1)$

(viii) $\ln/\sin ax/ + C$

(ix) $\frac{-1}{7} \frac{\cos^7 ax}{a}$

(x) $(+ \tan x)^2 + C$

3.2 INTEGRATION OF INVERSE TRIGONOMETRIC FUNCTION

Recall that $\frac{d}{dx}(\text{arc sin } u) = \frac{1}{\sqrt{1-x^2}\_dx} \quad \underline{du}$

to evaluate $\int$arc sin u du you have to know how to integrate by part which is one of the techniques of integration that you will study next unit. For now
$\int \text{arc sin } u \, du = u \text{ arc sin } u + \sqrt{1 - u^2} + C$ and
$\int \text{arc tan } u \, du = u \text{ ar tan } u = - \frac{1}{2} \ln/1+u^2/+C$

You can proceed to make use of the derivative of arc tan x to evaluate special integrals.

Recall $\frac{d}{du}(\text{arc tan } u) = \frac{1}{1+u^2}$
$u^2 = a^2v^2$
therefore: $\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{adv}{\sqrt{a^2-a^2v^2}} = \int \frac{adv}{a\sqrt{1-v^2}}$

$= \int \frac{dv}{\sqrt{1-v^2}} = \text{arc sin } v + C$
$= \text{arc sin } \frac{u}{a} + C$

Example: Find $\int \frac{du}{\sqrt{4-u^2}}$

Solution: $\int \frac{du}{\sqrt{4-u^2}} = \int \frac{du}{\sqrt{(2)^2- u^2}} = \text{arc sin } \frac{u}{} + C = \text{arc sin } \frac{u}{2} + C$

Example: Find (1) $\int \frac{dx}{a^2+(x+2)^2}$

Solution let $u = (x + 2)$, $du = dx$

therefore $\frac{dx}{a^2 + (x + 2)^2} = \int \frac{du}{a^2 + u^2}$

$$= \frac{1}{a} \arctan \frac{u}{a} + C$$

$$= \frac{1}{a} \arctan \frac{(x+2)}{a} + C$$

Example: Find $\int \frac{dx}{\sqrt{a^2 + (x-1)^2}}$

Let $u = x-1$ $du = dx$

therefore $\int \frac{dx}{\sqrt{a^2+u^2}} = \arcsin \frac{u}{a} + C$

$$= \arcsin \frac{x-1}{a} + C$$

Exercises: Find the following integrals:

(i) $\int \frac{dx}{16 + 4x^2}$ (ii) $\int \frac{dx}{\sqrt{9 - 64x^2}}$

(iii) $\int \frac{dx}{49 + (x+2)^2}$ (iv) $\int \frac{dx}{\sqrt{25 - 9x^2}}$

(v) $\int_0^5 \frac{dx}{25+x^2}$

Ans: (i) $\frac{1}{4} \arctan \frac{x}{8}$ (ii) $\arcsin \frac{8x}{3}$

(iii) 1/7 art tan $\frac{x+2}{7}$ (iv) $\arcsin \frac{3x}{5}$

(v) $\prod/20$

4.0 CONCLUSION

In this unit you have derived the formula for common rational functions and how to find their integrals. You studied how to derive the integration formula of trigonometric functions. Evaluation and trigonometric functions were treated. You also find the integrals of special functions using the inverse functions of sin x and tan x. The formulas derived in this unit will be used to study methods and techniques of integration which will be studied in the next unit of this course.

5.0 SUMMARY:

In this unit you have studied how to;

1. derive formula such as:

   (i) $\int \frac{1}{u} du = \ln/u/+C$
   (ii) $\int \sin u \, du = -\cos u + C$
   (iii) $\int \cos du = \sin u + C$
   (iv) $\int \tan u \, du = \ln/\sec u/+ C$
   (v) $\int \cot u \, du = \ln/\sin u/+C$
   (vi) $\int \sec u \, du = \ln/\tan u + \sec u/$
   (vii) $\int \operatorname{cosec} u \, du = \ln/\operatorname{cosec} u - \cot u/ + C$
   (viii) $\int e^u du = e^u + C$

2. evaluate integral of this form $\int \frac{du}{\sqrt{a^2-u^2}} = \text{arc} \sin \frac{u}{a} + C$
   and $\int \frac{du}{a^2+u^2} = \frac{1}{a} \text{arc} \tan \frac{u}{a} + C$

3. how to use the formula in (i) above to evaluate integrals.

6.0 TUTOR MARKED ASSIGNMENT

Find the following integrals

(1) $\int \frac{dx}{5 - 7x}$ (2) $\int \frac{1}{x - 6} dx$ (3) $\int \frac{x \, dx}{x^2 - 4}$

(4) $\int \frac{10x + 5}{5x^2+5x+1} dx$ (5) $\int e^4 dx$ (6) $\int \sin(4x-1) dx$

(7) $\int \sin^c x \, \text{cov}x \, du$ (8) $\int \frac{du}{\sin^2 x}$ (9) $\int \sin^4 ax \cos ax \, dx$

(10) $\int x \cot(x)^2 dx$ (11) $\int \frac{du}{16 + x^2}$ (12) $\int \frac{dx}{\sqrt{90^2 - 4x^2}}$

(13) $\int 4x^3 e^{x4} dx$ (14) $\int (e^x + x)^2 (e^x + 1) dx$

(15) $\int \cos 2x \sin 2x \, dx$ (16) $\int \frac{dx}{\sqrt{36-(x+3)^2}}$ (17) $\int 3\tan (x+1)^2 dx$

(18) $\int xe^{x2}dx$ (19) $\int \cos^8 x \sin x \, dx$ (20) $\int \frac{3x^2}{x^3 - 8} dx$

UNIT 5

INTEGRATION OF POWERS OF TRIGONOMETRIC FUNCTIONS

TABLE OF CONTENTS

1.0 INTRODUCTION

So far what you have studied in the last two units is to find the function whose derivative gives you the integral of another function. This process is summed up in the fundamental theorem of integral calculus. For a review, consider evaluating the integral $\int f(x)dx$ what you have studied in unit 2 and 3 is to find a function F(x) such that $d/dxF(x) = f(x)$ – 1 then $F(x)+C = \int f(x)dx$.

The process of finding F(x) that satisfies equation 1 above is the difficult aspect and that is why differentiation is taught before integration. So far, all you have been doing is making a good guess for the function F(x) which is dependent on how familiar you are with differentials of functions. In this unit you will study how to make the guesswork a lot easier. This will be done by introducing firstly the use of differentiation formulas along side their integration formulas, second, by applying some techniques that will be developed here based on the knowledge of function as well as t heir respective derivative. Since it is the anti-derivative that gives the solution to the integral it is necessary once again you review basic rules and formulas for derivatives of function in the course calculus I.

The emphasis in this unit would be on developing skills rather than finding specific answer to any given problem. Therefore as was done in the previous units a particular example might be solved several times with different methods. Therefore the examples in this unit have been kept fairly simple so that you would be able to develop the necessary skills expected of your.

2.0 OBJECTIVES

After studying this unit you should be able to correctly

1. Recall differential formulas and their corresponding integrals
2. Evaluate integrals involving powers of trigonometric functions
3. Evaluate integrals involving products of even powers of sines and cosines
4. to develop techniques and methods for evaluating integrals of any function formed by functions of the trigonometric functions.

3.0 BASIC FORMULA

The first requirement for skill in integration is a thorough mastery of the formulas for differentiation. Therefore, a good starting point for you to develop the skill required of you in this course is for you to build your own table of integral. You may make your own note in which the various sections are headed by standard form like $SU^n du$ and then under each heading include several examples to illustrate the range of application of the particular formula. Therefore, what will be done in this unit is to list formulas for differentiation together with their integration counterparts.

Summary of Differential Formulas and Corresponding Integrals

1.	$du = \underline{du}\ dx$	1	$\int du = u\ C$
2.	$d(au) = a\,du$	2	$\int a\,du = a \int du$
3.	$d(u + v) = du + dv$	3.	$\int (du + dv) = \int du + \int dv$
4..	$d(u^n) = nu^{n-1}du$	4.	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq 1$
5.	$d(\ln u) = \underline{du}$	5.	$\int \underline{du} = \ln /u/ + C$
6.	a) $d(e^u) = e^u\,du$	6.	a) $\int e^u\,du = e^u + C$
	b) $d(a^u) = a^u \ln a\,du$		b) $\int a^u\,du = \frac{a^u}{\ln a} + C$
7.	$d(\sin u) = -\sin u\,du$	7.	$\int \cos u\,du = \sin u + C$
8.	$d(\cos u) = -\sin u\,du$	8.	$\int \sin u\,du = \tan u + C$
9.	$d(\tan u) = \sec u \tan u\,du$	9.	$\int \sec^3 u\,du = \tan u + C$

10. $d(\cot u) = -\csc^2 u\, du$ 10. $\int \csc^2 u\, du = -\cot u + C$

11. $d(\sec u) = \sec u \tan u\, du$ 11. $\int \sec u \tan u\, du = \sec u + C$

12. $d(\csc u) = -\csc u \cot u\, du$ 12. $\int \csc u \cot u\, du = -\csc u + C$

13. $d(\sin^{-1} u) = \dfrac{du}{\sqrt{1 - u^2}}$ 13. and $\int \dfrac{du}{\sqrt{1 - u^2}} = \begin{cases} \sin^{-1} u + C \\ -\cos^{-1} u + C \end{cases}$

14. $d(\cos^{-1} u) = \dfrac{du}{\sqrt{1 - u^2}}$ 14.

15. $d(\tan^{-1} u) = \dfrac{du}{1 + u^2}$ 15. and $\int \dfrac{du}{1 + u^2} = \begin{cases} \tan^{-1} u + C \\ -\cot^{-1} u + C \end{cases}$

16. $d(\cot^{-1} u) = \dfrac{-du}{1+ u^2}$ 16.

17. $d(\sec^{-1} u) = \dfrac{du}{/u/ \sqrt{u^2 - 1}}$ 17. and $\int \dfrac{du}{u\sqrt{u^2 - 1}} = \begin{cases} \sec^{-1/u/} + C \\ -\csc^{-1}/u/ + C \end{cases}$

18. $d(\csc^{-1} u) = \dfrac{-du}{/u/\sqrt{u^2 - 1}}$ 18.

3.1 INTEGRATION INVOLVING POWERS OF TRIGONOMETRIC FUNCTIONS

From the above basic formula you have that:

(1) $\int u^n\, du = \dfrac{u^{n+1} + C}{n + 1}$ for $n \neq -1$

and

(2) $\int u^n\, du = \ln/u/+C \quad n = 1$

This could be used to evaluate integrals involving powers of trigonometric functions.

Example: Find $\int \sin^n ax \cos ax\, dx$

Let $u = \sin ax \; du = a\cos ax\, dx$

then $\dfrac{du}{a} = \cos ax\, dx$, $u^n = \sin^n ax$

therefore: $\int \sin^n ax \cos ax\, dx = \int u^n \dfrac{du}{a}$

$= \underline{n^{n+}}1 + C$

$$a(n + 1)$$

using equation (1) above you get

(3) $\int \sin^n ax \cos ax\, dx = \dfrac{\sin^{n+1} ax}{a(n + 1)} + C$

with equation (2) you get n + 1

(4) $\int \dfrac{\cos ax\, dx}{\sin ax} = \dfrac{1}{a} \ln /\sin ax/ + C$

Interestingly this is the same result arrive at when you derive the formula for $\int \cot u\, du = \int \dfrac{\cos ax}{\sin ax} dx = \ln/\sin u/ + C$

In a similar manner you can find $\int \cos^n ax \sin ax\, dx$

Let $u = \cos ax$ $du = -a \sin ax$

$U^n = \cos^n ax$ then

$\int \cos^n ax \sin ax\, dx = \int u^u(\underline{-du}) = -\dfrac{U^{n+1} C}{n + 1}$

for $n \neq 1$

therefore $\int \cos^n ax \sin ax\, dx = \dfrac{-\cos^{n+1} ax}{(n + 1)a}\ C$

for n = 1 $\int \dfrac{\sin ax}{\cos ax} dx\ \dfrac{-1}{a} \ln/\cos ax/ + C$

this is the same as $\int \tan ax\, dx$

i.e. $\int \tan ax\, dx = \underline{-1.} \ln/\cos ax/ + C$

$= \dfrac{1}{a} \ln/\sec ax/ + C$

(see 3.2 of Unit 4)

Example: Try finding $\int \sin^3 x\, dx$ you find out that the above method does not work because there is cos x side of it to give d (sin x)/ therefore, another method has to be tried.

Recall that $\sin^3 x = \sin^2 x \sin x$

$= (1 - \cos^2 x) \sin x$

$\sin^3 x = \sin x - \cos^2 x \sin x$

then let $u = \cos x$ $du = -\sin x$

$\int \sin^3 dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + C$

The above give rise to a formula or technique for integrating odd powers of sin x or cos x

i.e. $\cos^{2n+1} x = \cos^{2n} x \cos x$
but $\cos^{2n} x = (\cos^2 x)^n = (1 - \sin^2 x)$
therefore $\cos^{2n+1} x = (1-\sin^2 x)^n . \cos x$
let u = sin x du = cos x dx
therefore $\int \cos^{2n+1} x\, dx = \int (1 - \sin^2 x)^n \cos x\, dx$
$= \int (1 - u^2)\, du.$

What follows next is to expand the expression $(1-u^2)$ du where u = cos x smf
$\int \cos^{2n+1} x\, dx = -\int (1-u^2)^n du$ where u = sin x

Example: Find (1) $\int \cos^3 x\, dx$ ii $\int \sin^5 x\, dx$

Solution: $\int \cos^{2n+1} x\, dx = \int (1 - u^2)^n du$ $2n + 1 = 3 \Rightarrow n = 1, u = \sin x$

therefore: $\int \cos^3 x\, dx = -\int (1 - u^2)\, du = u - \frac{u^3}{3}$

$= \sin x - \frac{\sin^3 x}{3} + C$

(optimal) $= \sin x - \frac{\sin x}{3} + \frac{\sin^2 x \cos x}{3}$

$= \frac{\sin^2 x \cos x}{3} - \frac{2}{3} \sin x$

(ii) $\int \sin^5 x\, dx$

$2n + 1 = 5 \Rightarrow$ n 2, u = cos x

therefore $\int \sin^5 x\, dx = \int (1 - u^2)^2 du$
$= \int (1 - 2u^2 + u^4)\, du$
$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$

therefore $\int \sin^5 x\, dx = \cos x - \frac{2}{3} \cos^3 + \frac{1}{5} \cos^5 x + C$

(optimal) $= \frac{1}{5} \cos^4 x \sin x + \frac{4}{15} \cos^2 x \sin x + \frac{8}{15} \sin x + C$

Example: Find $\int$ secx tan x dx

Solution: $\int \sec x \tan x\, dx = \int \frac{1}{\text{Cos } x} \frac{\sin x}{\cos x} = \int \frac{\sin x}{\cos^2 x}$

then $\int \sec x \tan x \, dx = \int \cos^{-2} x \sin x \, dx$

$\Rightarrow$ therefore $\int \cos^{-2} x \sin x = \dfrac{-\cos^{-2+1}}{-2+1} + C$

$= \dfrac{\cos^{-1}}{-1} x + C$

$= \dfrac{-1}{\cos^{x}} + C$

$= \sec x + C$

Example: Find $\int \tan^4 x \, dx$

recall that $\sin^2 + \cos^2 x = 1$

therefore $\tan^2 x = \sec^2 x - 1$

then

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x . \tan^2 x \, dx \\ &= \int \tan^2 x (\sec^2 x - 1) \, dx \\ &= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx \\ &= \int \tan^2 x \sec^2 x - \int (\sec^2 x - 1) \, dx \\ &= \int (\tan^2 x \sec^2 x) \, dx - \int \sec^2 x \, dx + \int dx\end{aligned}$$

let $u = \tan x \quad du = \sec^2 x \, dx$

$$\begin{aligned}\text{therefore } \int \tan^4 x \, dx &= \int u^2 \, du - \int du - \int dx \\ &= \frac{u^3}{3} - u - x \\ &= \frac{1}{3} \tan^3 x - \tan x + x + C\end{aligned}$$

Therefore, for n = even you can derive the formula using the technique above.

$$\begin{aligned}\int \tan^n x \, dx &= \int \tan^{n-1} x (\sec^2 x - 1) \, dx \\ &= \int \tan^{n-2} x \sec^2 x) \, dx - \int \tan^{n-2} x \, dx \\ &= \int (\tan^{n-2} x \sec^2 x) \, dx - \int (\sec^{n-2} x - 1) \, dx \\ &= \int (\tan^{n-2} x \sec^2 x) \, dx - \int \sec^{n-2} + dx \\ &= \int \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x \, dx\end{aligned}$$

Example: Find $\int \tan^2 x \, dx$

$n = 2$, therefore $n - 1 \Rightarrow 2 - 1 = 1$

$$\begin{aligned}\text{therefore } \int \tan^2 x \, dx &= \frac{\tan x}{1} - \int \tan^0 x \, dx \\ &= \tan x - x\end{aligned}$$

The above formula also works for the case n is odd. Let $n = 2m + 1$ then after m steps it will be reduced by 2m leaving $\int \tan x = -\ln/\cos x/ + C$.

From the two examples above, you can see the usefulness of the two trigonometric identities.

$\sin^2 x + \cos^2 x = 1$ and
$\tan^2 x + 1 = \sec^2 x$

in evaluating integrals involving powers of trigonometric functions such as

(a) odd powers of sin x or cos x
(b) any integral powers of tan x (or cot x) and
(c) even powers of sec x (cos x u)

To get the integral C of even powers of sec x all you need do is to express $\sec^2 x$ in terms of $\tan^2 x$ and then use the reduction process above to get the integral.

Example: Find $\int \sec^4 x\, dx$

$$= \int \sec^2 dx \sec^2 x = \int \sec^2 x (1 + \tan^2 x)\, dx$$
$$= \int \sec^2 dx + \int \tan^2 x \sec^2 x\, dx$$
$$= \int /+ \tan^2 x\, dx + \int u^2\, du$$

where u = tan x and du = $\sec^2 x$ dx.

$$\Longrightarrow \int u^2\, du = \frac{u^3}{3} + C$$

but $\int \tan^2 x\, dx = \tan x - x$

$$\int \sec^u x\, dx = \int dx + \int \tan^2 x\, dx + \frac{\tan^3 x}{3} + C$$

You can now derive the integral for any even powers of sec x

Example: Find $\int \sec^n x\, dx$

Solution: let $\int \sec^2 x\, dx = \int (\sec^{2n-2} x)(\sec^2 x)$

$$= \int (\sec x)^{2(n-1)} \sec^2 x\, dx$$

$$= \int (\sec^2 x)^{n-1} \sec^2 x\, dx$$

$$= \int (1 + \tan^2 x)^{n-1} \sec^2 x\, dx$$

$= \int (1 + u^2)^{n-1}\, du$ (where u = tan x and du = $\sec^2 x$ dx) where $(1 + u^2)^{n-1}$ can be expanded by the binomial theorem and then the result will be integrated term by term as;

Exercises:

(i) $\int \sin^3 x\, dx$ (ii) $\int \tan^2 4x\, dx$ (iii) $\int \cos^5 x\, dx$

(iv) $\int \cot^3 x\, dx$ (v) $\int \cos^3 x \sin^2 x\, dx$ (vi) $\int \sec^u x \tan u\, du$

(vii) $\int \frac{dx}{\sin x}$ (viii) $\int \cos^n x \sin x\, dx$ (ix) $\int \cos^2 x \sin 2x\, dx$

(x) $\int \cos x^4\, 3x\, dx$

Ans:

(i) $1/3 \cos^3 x - \cos x + C$

(ii) $\tan^4 x - 4x + C$

(iii) $\sin x - \frac{2}{3}\sin^3 x + 1/5 \sin^5 x + C$

(iv) $\frac{-\cot^2 x}{2} - \ln/\sin x/ + C$

(v) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(vi) $\ln(\text{cosec } x - \cot x) + C$

(vii) $\ln(\text{cosec } x - \cot x) + C_2$

(viii) $\frac{-\cos^{n+1} x}{n+1} + C$

(ix) $\frac{-\cos^3 2x}{6} + C$

(x) $\frac{-1}{9}\text{cosec}^2\, 3x \cot 3x\, \frac{-2}{9} \cot 3x$

7.2 INTEGRATION OF EVEN POWERS OF SINES AND COSINES

In the previous section you have studied how to integrate odd powers of sin x and cos x. You will attempt to evaluate integrals of even powers of sines and cosines by applying the same technique used above for odd powers i.e.

$\int \sin^n x \cos^m x\, dx$ where m or n is an even numbers.

that $\int \cos^{1/2} x \sin^3 dx$ evaluate the integral.

Recall that 3 is odd as such $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$.

therefore: $\int \cos^{1/2} x \sin^3 x\, dx = \int \cos^{1/2} x\, 91 - \cos^2 x) \sin x\, dx$

for u = cos x du = sin x dx.

therefore: $\int \cos \frac{1}{2} x (1 - \cos^{12} x) \sin dx = \int u^{1/2} (1 - u^2)\, du$

$= \int (u^{1/2} - u)\, du = 2\frac{u^{3/2}}{3} - \frac{u^2}{2}$

$= \frac{2\cos^{3/2} x}{3} = \frac{\cos^2 x}{2} + C$

3 2

If in the above you have $\sin^4 x$ instead of $\sin^3 x$ then you have to evaluate

$\int \cos^{1/2} x \sin^4 x \, dx$

Then using the above method will fail because $\sin^4 x = (1 - \cos^2 x)^2$ which give

$$\int \cos^{1/2} x \sin^4 x \, dx = \int \cos^{1/2} x (1-\cos^2 x)^2 \, dx$$

missing above is $-\sin x \, dx = du$ that goes with the cos x. Therefore, there is a need to use another trigonometric identity. The one that will be used is given as $\sin^2 x = ½ (1 - \cos 2x)$ and $\cos^2 x = ½ (1 + \cos 2x)$.

Note: The above identities are derived by adding or subtracting the equations $\cos^2 x + \sin^2 x = 1$ and $\cos^2 x - \sin^2 x = \cos 2x$

Recall

$$\begin{aligned}\int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx \\ &= \int [½ (1 - \cos 2x]^2 \, dx \\ &= \int ¼ (1 - 2 \cos 2x + \cos^2 2x) \, dx \\ &= ¼ \int (1 - 2 \cos 2x + ½ (1 - \cos 4x) \, dx \\ &= ¼ [x - \sin 2x + \frac{x}{2} - \frac{1}{8} \sin 4x] \\ &= \frac{3}{8} x - ¼ \sin 2x = \frac{1}{32} \sin 4x + C\end{aligned}$$

Example: Find $\int \sin^2 x \cos^2 x \, dx$

Here both powers are even.

Let $\sin^2 x = (1 - \cos^2 x)$

Therefore $\sin^2 x \cos^2 x = (1 - \cos^2 x) \cos^2 x$

$\Rightarrow \int \sin^2 x \cos^2 x \, dx = \int (\cos^1 x - \cos^4 x) \, dx$

$\int \cos^2 x \, dx - \int \cos^4 x \, dx$

$\int \cos^2 x \, dx = \int ½ (1 + \cos 2x) \, dx = \frac{x}{2} + \frac{\sin 2x}{4}$

$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int [½ (1 + \cos 2x)]^2 \, dx$

$= \int ¼ [1 + 2 \cos 2x + \cos^2 2x] \, dx$

$= ¼ [1 + 2\cos 2x + ½ (1 + \cos 4x)] \, dx$

$$= \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

$$\int \sin^2 x \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

$$= \frac{7}{8}x \quad \frac{1}{2}\sin 2x + \frac{1}{32}\sin 4x + C$$

Example: Find $\int \cos^6 x\, dx$

$\Longrightarrow \int \cos^6 x\, dx = \int(\cos^2 x)^3\, dx = \int \frac{1}{8}(1 + \cos 2x)^3\, dx$

$$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\, dx$$

$$= \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x \frac{-1}{48}\sin^3 2x + C.$$

Exercises: Find the following integrals:

(i) $\int \sin^2 x \cos^4 x\, dx$ (ii) $\int \sin^2 4t\, dt$

(iii) $\int \cos^2 6x\, dx$ (iv) $\int \sin^6 x\, dx$

(vi) $\int \cos^4 ax\, dx$

Ans:

(i) $\frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$ (ii) $\frac{x}{2} - \frac{\sin 8x}{16} + C$

(iii) $\frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$

(iv) $\frac{5}{16}x - \frac{1}{4}\sin 2x - \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

(v) $\frac{3}{8}x + \frac{1}{4}\sin 2ax + \frac{1}{32}\sin 4ax + C$

3.3 POWERS AND PRODUCTS OF OTHER TRIGONOMETRIC FUNCTIONS

In this section, you shall evaluate two types of integrals

(1) $\int \tan^m x \sec^n x\, dx$ and

(2) $\int \cot^m x\ \mathrm{cosex}^n\, dx$

Example: When n is even you write $\tan^m x \sec^n x = \tan^m x \sec^{n-2} x \sec^2 x$ and then express $\sec^{n-2}$ in terms of $\tan^2 x$ using $\sec^2 x + 1 = \tan^2 x$.

Example: $\int \tan^3 x \sec^2 x\, dx$

let $u = \tan x$ $du = \sec^2 x\, dx$.

then $\int \tan^3 x \sec^2 x\, dx = \int u^3\, du$

$$= \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$$

When n and m are both odd you write

$\tan x \sec^n x = \tan^{m-1} x \sec^{n-1} \sec x$ 1an u

and express $\tan^{m-1} x$ in terms of $\sec^2 x$ using $\tan^2 x = \sec^2 x - 1$

Example: $\int \tan^3 x \sec^3 x\, dx$

$\tan^3 x \sec^3 x = \tan^2 x \sec^2 x \tan x \sec x$

and $\tan^2 x = (\sec^2 x - 1)$

therefore: $\int \tan^3 x \sec^3 x\, dx = \int (\sec 3x - 1) \sec^2 x \sec x \tan x\, dx$

$$= \int (\sec^4 x - \sec^2 x) \sec x \tan x\, dx$$

(but u = sec x, du = se x 1 an x du)

therefore $\int \tan^3 x \sec^3 x\, dx = \int (u^4 - u^2)\, du$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^3 x}{5} - \frac{\sec^3 x}{3} + C$$

you can do the same for $\cot^m x\ \mathrm{cosec}^n x$ in a similar manner. That is for $\int \cot^m x\ \mathrm{cosec}^n x\, dx$ when n is even you write out $\cot^m x\ \mathrm{cosec}^n x = \cot^m x\ \mathrm{cosec}^{n-2}$ cosec x and express cosec^{n-2} in terms of $\cot^2 x$ using

$\text{cosec}^2 x = \cot^2 x + 1$

Example: $\int \cot^5 x \text{ cosec}^4 x \, dx$

$= \int \cot^5 x \text{ cosec}^2 x \text{ cosec}^2 x \, dx$
$= \int \cot^5 x (\cot^2 x + 1) \text{ cosec2 } x \, dx$
$= \int \cot^7 \text{cosec}^2 x \, dx + \int \cot^5 x \text{ cosec}^2 x \, dx$
$(u = \cot x \; du = -\text{cosec}^2 x \, dx).$
$= \frac{-\cot^8 x}{8} - \frac{-\cot^6 x}{6} + C$

In similar manner when m and n are both odd you have $\cot^m x \text{ cosex}^n x = \cot^{m-1} x \text{ cosec}^{n-1} x \text{ cosec } x \cot x$ and then express $\cot^{m-1} x$ in terms of $\text{cosec}^2 x$ using $\cot^2 x = \text{cosec}^2 x - 1$

Example: $\int \cot^5 x \text{ cosec}^3 x \, dx$

$= \int \cot^4 x \text{ cosec}^2 x \text{ cosec } x \cot x \, dx$
$= \int (\text{cosec}^2 x - 1)^2 \text{ cosec}^2 x \text{ cosec } x \cot x \, dx$
$= \int (\text{cosec}^6 x - 2 \text{ cosec}^4 x + \text{cosec}^2 x) \text{ cosec } x \cot x \, dx$
$u = \text{cosec } x \; du = - \text{cosec } x \cot x \, dx$

$= \int (u^6 - 2u^4 + u^2) (-du)$
$= \frac{-u^7}{7} + 2\frac{u^5}{5} - \frac{u^3}{3} + C$
$= \frac{-\text{cosec}^7}{7} x + \frac{2\text{cosec}^5}{5} x - \frac{\text{cosec}^3}{3} x + C$

Exercises Find

1. $\int \cot^3 x \text{ cosec}^3 x \, dx$
2. $\int \cot^3 x \text{ cosec}^2 x \, dx$
3. $\int \tan^5 x \sec^2 x \, dx$

Ans:

1. $\frac{-\text{cosec}^5 x}{5} + \frac{\text{cosec}^3}{3} x + C$
2. $\frac{-\cot^4 x}{4} + C$
3. $\frac{\tan^6 x}{6} + C$

4.0 CONCLUSION

In this unit, you have reviewed differential formulas and their corresponding integrals. These basic formulas will be used throughout the remaining part of the course. You have developed techniques of finding integrals of powers of trigonometric functions by using the trigonometric identities;

(i) $\cos^2 x + \sin^2 x = 1$ and
(ii) $1 + \tan^2 x = \sec^2 x$ etc.

You have also studied how to evaluate the products of even powers of sines and cosines functions. These integrals will be used when developing other techniques of integration in the next unit of this course.

5.0 SUMMARY:

You have studied in the unit how to

(i) recall basic differential formulas and corresponding integrals
(ii) use these basic formulas to develop techniques of integration of powers of trigonometric function
(iii) evaluate the integrals of odd powers of trigonometric function such as $\int \sin^n x\,dx$, $\int \cos^n x\,dx$
(iv) evaluate the integrals of trigonometric function such as $\int \tan^n x\,dx$, $\int \cot^n x\,dx$ where n is odd or even
(v) evaluate the integrals of even powers of sec x and cosec x
(vi) evaluate the integrals of products of even powers of sin x and cos x such as $\int \cos^n x\,dx$, $\int \sin^n x\,dx$, $\int \cos^n x\,dx \sin^m x\,dx$ where n or m is even or both are even.

6.0 TUTOR MARKED ASSIGNMENT

(1) Find $\int \sin^2 x \cos^2 x\,dx$
(2) Show that $\int \tan ax\,dx = \frac{1}{a} \ln/\cos ax/+C$
(3) Find $\int \sin^3 4x\,dx$
(4) Find $\int \tan^5 x \sec^3 x\,dx$
(5) Show that $\int \sec^{2n} x\,dx = \int(1+u^2)^{n-1}\,du$ where $u = \tan x$
(6) Find $\int \cos^{2/3} x \sin^5 x\,dx$
(7) Find $\int \sin^2 x \cos^5 x\,dx$
(8) Find $\int \sin 4x \cos^2 x\,dx$
(9) Find $\int \tan^6 x\,dx$
(10) Find $\int \tan^5 x \sec^4 x\,dx$

UNIT 6

FURTHER TECHNIQUES OF INTEGRATION I

TABLE OF CONTENTS

1.0 INTRODUCTION

In continuation of development of skills in techniques for finding integrals of special functions, you will study in the unit how to evaluate integrals involving rational function with $\sqrt{a^2-u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2- a^2}$ and $a^2 – u^2$ as denominators. The use of inverse trigonometric functions or trigonometric identifies will be needed. In this unit the method or process used in deriving formulas for integrals of functions in the previous unit will be adopted.

2.0 OBJECTIVES

In this unit you should be able to correctly;

(i) recall basic differential formulas and corresponding integrals as stated in UNIT 5.
(ii) Use these basic formulas to evaluate integrals involving $\sqrt{a^2-u^2}$, $a^2 + u^2$, u^2, - a^2, and $a^2 – u^2$

(iii) evaluate integrals of rational functions with $ax^2 + b\ x + C$ as denominator.

3.0 INTEGRALS INVOLVING $\sqrt{a^2 \neq u^2}$ and $a^2 \neq u^2$

Recall that $\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1 + u^2}\frac{du}{dx}$

therefore $\int \frac{d}{d}(\text{arc tan } u) = \int \frac{1}{1+u^2}\ du$

therefore arc tan u + C = $\frac{1}{1+u^2}$ – (I)

Example: Find the integral of $\frac{1}{a^2 + u^2}$ i.e. $\int \frac{du}{a^2 + u^2}$

To evaluate the above you factor out a^2 from $a^2 + u^2$
i.e. $a^2 + u^2 = a^2 (1 + (\frac{u}{a})^2)$

let $z = \frac{u}{a}$, $adz = du$

then $a^2 + u^2 = a^2(1 + z^2)$

therefore: $\int \frac{du}{a^2+u^2} = \frac{1}{a^2} \int \frac{adz}{1+z^2} = \frac{1}{a} \int \frac{dz}{1+z^2}$

$= \frac{1}{a} \tan^{-1} z$

thus $\int \frac{du}{a^2+u^2} = \frac{1}{a} \text{arc tan } \frac{u}{a} + C$ – II

Example: Find $\int \frac{du}{9 + u^2}$

Solution $\int \frac{du}{a+u} = \int \frac{du}{(3)^2+u^2} = \frac{1}{3} \text{arc tan } \frac{u}{3} + C$

You have to review some trigonometric identities you studied in MATH 111.

Example:

(i) $1 - \sin^2 x = \cos^2 x$
(ii) $1 + \tan^2 x = \sec^2 x$ and
(iii) $\sec^2 x - 1 = \tan2 x$.

let $u = a \sin x$
then $u^2 = a^2 \sin^2 x$

Multiplying identity (1) through by a^2 you get:

$a^2(1-\sin^2 n) = a^2 \cos^2 x$. – (i)

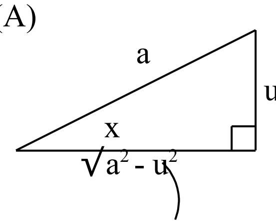
$a^2 - a^2 \sin^2 x = a^2 \cos^2 x$. – (ii) but $u^2 = a^2 \sin^2 x$ therefore equation (II) becomes $a^2 - u^2 = a^2 \cos^2 x$

In similar manner, if $a^2 + a^2 \tan^2 x = a^2 \sec^2 x$ and $u^2 = a^2 + ax^2 x$ then
$a^2 + a^2 \tan^2 x = a^2 + u^2 = a^2 \sec^2 x$.
If $a^2 \sec^2 x - a^2 = a^2 \tan^2 x$ and $u^2 = a^2 \sec^2 x$
then $u^2 - a^2 = a^2 \tan^2 x$.
thus:

(1) $a^2 - u^2 = a^2 \cos^2 x$,; $u = a \sin x$ (see fig. 6.1)
(2) $a^2 + u^2 = a^2 \sec^2 x$; $u = a \tan x$ (see fig. 6.2)
(3) $u^2 - a^2 = a^2 \tan^2 x$; $u = a \sec x$ (see fig. 6.3)

The above trigonometric identities you were given in equations (1) to (3) are equivalent expressions of the Pythagorean Theorem. See fig. 6.1 to 6.3

(A)

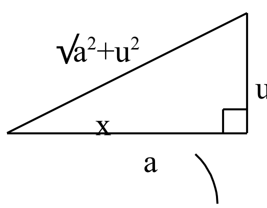


$\frac{u}{a} = \sin x$

and $\sqrt{a^2-u^2} = a \cos x$

⟹ $a^2 - u^2 = a^2 \cos x$

Fig. 6.1



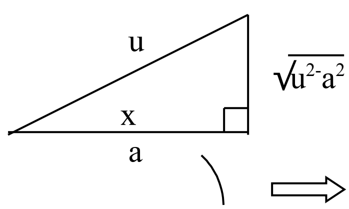
$\frac{u}{a} = \tan x$

and $\frac{u}{\sqrt{a^2+u^2}} = \cos x$

⟹ $= a \sec x$

$= a^2 \sec^2 x$

Fig. 6.2



$\frac{u}{a} = \sec x$

$\frac{\sqrt{u^2-a^2}}{a} = \tan x$

⟹ $u^2 - a^2 = a^2 \tan^2 x$

Fig. 6.3

Example Find $\int \frac{du}{\sqrt{a^2-u^2}}$ $/a/>u$

Solution: let $u = a \sin x \Longrightarrow du = a \cos x\, dx$
then $a^2 - u^2 = a^2 \cos^2 x$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a \cos x\, dx}{\sqrt{a^2 \cos^2 x}} = \int dx = x + C$$

if $u = a \sin x \Longrightarrow \frac{u}{a} = \sin x$

and $x = \arc \sin \frac{u}{a}$

therefore $\int \frac{du}{\sqrt{a^2-u^2}} = \arc \sin \frac{u}{a} + C$ - III

(B) The usefulness of trigonometric identities in evaluating special types of integrals is numerous. Functions not involving trigonometric function can be integrated by expressing them in terms of trigonometric identities and then using standard integration formulas to evaluate them. In integration by method of completing the square is introduced in this unit.

Example: Find $\int \frac{du}{\sqrt{25-u^2}}$

here $a^2 = 25 \Longrightarrow a = 5$

From above $u = 5 \sin x \Longrightarrow du = 5 \cos x\, dx$

$$\int \frac{du}{\sqrt{25 - u^2}} = \int \frac{5 \cos x\, dx}{\sqrt{25 \cos^2 x}} = \int dx = x + C$$

therefore $x = \arc \sin \frac{u}{5} + C$

Example: Find $\int \frac{du}{\sqrt{a^2+u^2}}$ $a > 0$

Let $u = a \tan$
$du = a \sec^2 x\, dx$
but $a^2 + u^2 = a^2 + a^2 \tan^2 x = a^2 (1 + \tan^2 x)$
$= a^2 \sec^2 x$

then $\int \frac{du}{\sqrt{a^2+u^2}} = \int \frac{a \sec^2 x\, dx}{\sqrt{a^2 \sec^2 x}} = \pm\int \sec x\, dx$
Recall that $\int \sec x\, dx = \ln/\sec x + \tan x/ + C^1$

i.e. $\int \frac{\sec x (\tan x + \sec x)}{\tan x + \sec x} dx = \ln/s\sec x + \tan x/ + C^1$

Hence ±∫ sec x dx = ln/sec + tan x/ + C^1
If you let x = arc tan $\underline{u}$ x∈ $(-^{\pi}/_2, {}^{\pi}/_2)$
then sec x will be positive and $\int \frac{du}{\sqrt{a^2 + u^2}} = \int \sec x\, dx$

$= \ln/\sec x + \tan x/ + C$

recall that you let $a^{2+u}2 = a^2 \sec^2 x$

⟹ $\sec x = \frac{\sqrt{a^2+u^2}}{a}$

and $\tan x = \frac{u}{a}$

Then

$$\int \frac{du}{\sqrt{a^2+u^2}} = \ln/\frac{\sqrt{a^2+u^2}}{a} + \frac{u}{a}/ + C^1$$

let $C = C^1 - \ln a$ you then have that

$$\int \frac{du}{\ } = \ln/\sqrt{a^2+u^2} + u/ + C. \_\_ \text{ IV}$$

Example: Find $\int \frac{du}{\sqrt{16 + u^2}}$

Solution: let a^2 16 ⟹ a = 4
then by direct substitution into IV you get

$$\int \frac{du}{\sqrt{16+u^2}} = \ln /\sqrt{16} + u^2 + u/ + C$$

Example: Find $\int \frac{du}{\sqrt{u^2 - a^2}}$ lu/>a>0

Solution: You can start by trying the substitution u = a sec x
then du = a sec x tan x dx
but $u^2 - a^2 = a^2 \sec^2 x - a = a^2 (\sec^2 x - 1)$
$= a^2 \tan^2 x$

You will then have that

$$\int \frac{du}{\sqrt{u^2-a^2}} = \int \frac{a \sec x \tan x\, dx}{\sqrt{a^2 \tan^2 x}} = \int \frac{a \sec x \tan x\, dx}{a \tan x}$$

$= \pm\int \sec x \, dx$

therefore $x = \text{arc sec } \frac{u}{a} \quad 0 < x < \pi$

but $\tan x > 0$ whenever $0 < x < {}^{\pi}/_{2}$.
and $\tan x < 0$ whenever $\pi/2 < x < \pi$

From the previous example, you know that $\pm\int \sec x \, dx = \ln/\sec x + \tan x/ + C^1$

Recall tha $\sec x = \frac{u}{a}$ and $\tan x = \pm\frac{\sqrt{u^2 - a2}}{a}$

If $\tan x > 0$ you get $+\ln/\frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a}/ + C^1$

and $\tan x < 0$ you get $-\ln/\frac{u}{a} - \frac{\sqrt{u^2 - a^2}}{a} / \ C^1$

However, $-\ln/\frac{u}{a} - \frac{\sqrt{u^2 - a^2}}{a}/ = \ln/\frac{u + \sqrt{a^2 + u^2}}{a}$

therefore $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln/u + \sqrt{u^2 - a^2}/ + C$ – (V)

(where $C = C^1 - \ln a$.)

Example Find $\int \frac{du}{\sqrt{u^2 - 64}}$

Solution: let $a^2 = 64 \Rightarrow a = 8$

Thus by direct substitution into equation (V) you get that

$\int \frac{du}{\sqrt{u^2 - 64}} = \ln/u + \sqrt{u^2 - 64}/ + C$

Example: Find $\int \sqrt{9 - u^2} \, du$

Solution: let $u = 3 \sin x$, $du = 3 \cos x \, dx$.
then $9 - u^2 = (9 - 9 \sin^2 x) = 9(1 - \sin^2 x)$
$= 9 \cos^2 x$

therefore: $\int \sqrt{9 - u^2} \, du = \int \sqrt{9 \cos^2 x} \;\; 3 \cos x \, dx$
$= \int 9 \cos^2 x \, dx$

From Unit 5 sec 3.3 you have that

$$\int 9\cos^2 x\, dx = 9/2 \int (1 + \cos 2x)\, dx$$
$$= 9/2\left(x + \frac{\sin 2x}{2}\right) + C$$
$$= 9/2\, x + \frac{9 \sin 2x}{4} + C$$

therefore: $\int \sqrt{9 - u^2}\, du = \frac{9}{2} \text{ arc sin } \frac{u}{3} \_ \frac{u}{2} \sqrt{9 - u^2} + C$

Example: Find $\int \frac{u^2\, du}{\sqrt{4 - u^2}}$

Solution: let $u = 2 \sin x$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$du = 2 \cos x\, dx$

$4 - u^2 = 4 - 4\sin^2 x = 4\cos^2 x$

therefore $\int \frac{u^2\, du}{\sqrt{4-u^2}} = \int \frac{4 \sin^2 x\; 2 \cos x\, dx}{\sqrt{4 \cos^2 x}} = \int 4 \sin^2 x\, dx$

From unit 5 sec 3.3 you have that

$$4 \qquad \int \sin^2 x\, dx = 4 \int \frac{(1 - \cos 2x)}{2}\, dx$$
$$= 2x - 2 \sin x \cos x + C$$
$$= 2(\text{arc sin } \frac{u}{2} - \left(\frac{4 - u^2}{4}\right) + C$$

Example: Find $\int \frac{dx}{\sqrt{1 - 4x^2}}$

Solution: let $2x = \sin u$, $4x^2 = \sin^2 u$

$2\, dx = \cos u\, du$

$1 - 4x^2 = 1 - \sin^2 u = \cos^2 u$

therefore: $\sqrt{1 - 4x^2} = \sqrt{\cos^2} u = \cos u$

hence $\int \frac{dx}{\sqrt{1-4x^2}} = ½ \int \frac{\cos u\, du}{\cos u} = ½ \int du$

$= ½\, u + C$ but $u = \text{arc sin } 2x$

$= \frac{1}{2} \arcsin 2x + C$

Example: Find $\int \frac{x\,dx}{\sqrt{4 + x^2}}$

Solution: let $x = 2 \tan u \quad dx = 2 \sec^2 u\, du$

$4 + x^2 = 4 + 2 \tan^2 u = 4 (1 + \tan^2 u) = 4 \sec^2 u$

$$\int \frac{x\,dx}{\sqrt{4+x^2}} = \int \frac{2\tan u . 2 \sec^2 u\, du}{\sqrt{4 \sec^2 u}} = \int 2 \tan u \sec u\, du$$

$$= \underline{2 \sec} u + C$$

$$= \sqrt{4 + x^2} + C$$

Example: Find $\int \frac{dx}{\sqrt{(x-1)^2 + 4}}$

Solution: let $z = \frac{x-1}{2}$, $2dz = dx$

$4 + (x-1)^2 = 4(1 + Z^2)$

$$\Rightarrow \int \frac{dx}{(x-1)^2 + 4} = \frac{1}{4} \int \frac{2\,dz}{1+Z^2} = \frac{1}{2} \int \frac{dz}{1 + Z^2}$$

$$= \frac{1}{2} \arctan z = \frac{1}{2} \arctan \frac{(x-1)}{2} + C$$

Exercises: Find the following integrals

(i) $\int \frac{dx}{(9 - x^2)^{3/2}}$ (ii) $\int \sqrt{16 + x^2}\, dx$

(iii) $\int \frac{dx}{x\sqrt{9x^2+4}}$ (iv) $\int \sqrt{25 - 4x^2}$

(v) $\int \frac{x^2\, dx}{\sqrt{9 - 4x^2}}$ (vi) $\int \frac{dx}{\sqrt{1-16x^2}}$

(vii) $\int \frac{\cos x\, dx}{\sqrt{2-\sin^2 x}}$ (viii) $\int \frac{dx}{\sqrt{1 - \frac{x^2}{16}}}$

(ix) $\int \frac{dx}{x\sqrt{a^2+x^2}}$ (x) $\int \frac{x\, dx}{\sqrt{25-4x^2}}$

Ans:

(i) $\underline{x\quad\quad} + C$ (ii) $\frac{1}{2} x \sqrt{16 + x^2} + 8\ln/x+\sqrt{16+x^2}/+C$

(iii) $\frac{1}{2} \ln\left|\frac{\sqrt{9x^2 + 4} - 2}{3x}\right| + C$

(iv) $\frac{x}{2}\sqrt{25 - 4x^2} + \frac{25}{4} \text{arc sin } \frac{2x}{5} + C$

(v) $\frac{-x}{8}\sqrt{9 - 4x^2} + \frac{9}{16} \text{arc sin } \frac{2x}{3} + C$

(vi) $\frac{1}{4}$ arc sin $4x + C$ (vii) arc sin $\left(\frac{\sqrt{2} \sin x}{2}\right) + C$

(viii) $\frac{x}{8}\sqrt{16 - x^2} + 2 \text{ arc sin } \frac{x}{4} + C$

(ix) $\frac{-1}{a} \ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right| + C$

(x) $- \frac{1}{4}\sqrt{25 - 4x^2}$

3.1 INTEGRAION BY COMPLETING THE SQUARE OF $ax^2 + bx + C$

Given a quadratic function of this form $f(x) = ax^2 + bx + C$ by completing the square it can be reduced to the form $a(u^2 + A)$

i.e. $ax^2 + bx + C = a(x^2 + \frac{b}{a}x) + C$

$= a(x^2 + \frac{bx}{a} + \frac{b^2}{4}) + C - \frac{b^2}{4a}$

$= a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$

if you let $u = xt \ \frac{b}{2a}$ and $A = \frac{4x-b^2}{4a^2}$

then $ax^2 + bx + C = a(u^2 + A)$.

When the integral involves the square root of $ax^2 + bx + C$ then you have to consider only the case for which $\sqrt{a(u^2+A)}$ will have only real roots.

Example: Find $\int \frac{dx}{\sqrt{x^2+2x}}$

Solution: $x^2+2x = \int\sqrt{(x+1)^2 - 1}$

$= \sqrt{u^2 - 1}$, $u = x + 1$; $du = dx$

then $\int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{du}{\sqrt{u^2 - 1}}$

$= \ln/u + \sqrt{u^2-1}/ + C$

$= \ln/(x+1) + \sqrt{x^2 + 2x}/ + C$

Example: $\int \frac{dx}{\sqrt{x^2 - 8x}}$

$\sqrt{x^2 - 8x} = \sqrt{(x-4)^2 - 16}$

let $u = x - 4$ and $du = dx$ then $\sqrt{x^2 = 8x} = \sqrt{u^2 - 16}$

therefore: $\int \frac{dx}{} = \ln/u \ \sqrt{u^2 - 16}/ \ _C$

$= \ln/x - 4 + \sqrt{x^2-8x}/ + C$

Example: Find $\int \frac{dx}{x^{2-}10x+29}$

Solution:

$x^2 - 10x + 29 = x^{2-}10x + 25 + 4 = (x-5)^2 + 2^2$

therefore: $\int \frac{dx}{x^2-10x + 29} = \int \frac{dx}{2^2 + (x-5)^2} = \int \frac{du}{2^2 + u^2}$

where $u = x - 5$, $du = dx$

thus $\int \frac{dx}{x^2-10x+29} = \int \frac{du}{2^2+ u^2} = \frac{1}{2} \arctan \underline{u} + C$

$= \frac{1}{2} \arctan \frac{(x-5)}{2} + C$

Example: Find $\int \frac{dx}{\sqrt{3-x^2+2x}}$

Solution: $\sqrt{3-x^2+2x} = \sqrt{-(x^2 - 2x) + 3} = \sqrt{-(x^2-2x+1)+1+3}$

$= \sqrt{-(x-1)^2 + 4} = \sqrt{4 - u^2}$

where $u = x - 1$ $du = dx$ then

$$\int \frac{dx}{\sqrt{3-x^2+2x}} = \int \frac{du}{\sqrt{4-u^2}} = \text{arc sin } \frac{u}{} + C$$

$$= \text{arc sin } \frac{x-1}{2} + C$$

Example: Find $\int \frac{dx}{4x^2+4x+10}$

Solution: $4x^2 + 4x + 10 = 4(x^2 + x) + 10$

$= 4(x^2 + x + ¼) + 10 – 4/4$

$= 4(x + ½)^2 + 9$

let $u = x + ½$ $du = dx$

then $\int \frac{dx}{4x^2+4x+10} = \int \frac{du}{4u^2 + 9}$

$= 1/6 \text{ arc tan } \frac{2u}{3} + C$

$= 1/6 \text{ arc tan } \frac{2x + 1}{3} + C$

4.0 CONCLUSION

In this unit, you have studied techniques used in evaluating integrals involving $\sqrt{a^2+u^2}$, $\sqrt{a^2 - u^2}$, $\sqrt{u^2 - a^2}$, $a^2 + u^2$, $ax^2 + bx + C$, and $\sqrt{ax^2+bx+C}$. You have used the trigonometric identities and formulas studied in unit 5 to develop the techniques for solving integrals involving the expressions mentioned above. In the next unit you will study other techniques of integration.

In this unit you have reviewed important trigonometric identities such as (i) $1 + \tan^2 x = \sec^2 x$, (ii) $1 – \cos^2 x = \sin^2 x$ and (iii) $\sec^2 x – 1 = \tan^2 x$. You have used the above identities to develop techniques for evaluating integrals involving $a^2 + u^2$, $\sqrt{a^2 – u^2}$, $\sqrt{u^2 – a^2}$ and $\sqrt{a^2 + u^2}$. You have also recall the method of completing the square of a quadratic function such as $f(x) = ax^2+bx+C$. You have used the method of completing the square to evaluate the integrals involving $ax^2 + bx + C$ and $\sqrt{ax^2 + bx + C}$. You used the formulas studied in unit 5 to evaluate the above mentioned integrals. In the next unit you will study other techniques for evaluating integrals.

5.0 SUMMARY:

In this unit you have studied;

(1) how to evaluate the following types of integrals

(i) $\int \frac{du}{a^2+u^2}$ (ii) $\int \frac{du}{\sqrt{a^2+v^2}}$

(iii) $\int \frac{du}{\sqrt{a^2-u^2}}$ (iv) $\int \frac{du}{\sqrt{u^2-a^2}}$

(2) how to evaluate integrals such as:

(i) $\int \frac{dx}{ax^2+bx+C}$ (ii) $\int \frac{dx}{\sqrt{ax^2+bx+C}}$

using the method of completing the square.

6.0 TUTOR MARKED ASSIGNMENT

Evaluate the following integrals:

1. $\int \frac{dx}{\sqrt{2x-x^2+3}}$
2. $\int \frac{x^2\,dx}{\sqrt{25-x^2}}$
3. $\int \sqrt{4-x^2}\,dx$
4. $\int \frac{du}{\sqrt{u^2-a^2}}$
5. $\int \frac{du}{}$ /a/>/u/ /u/>/1/>0
6. $\int \frac{du}{u\sqrt{9u^2+4}}$
7. $\int \frac{dx}{x^2+2x+5}$
8. $\int \frac{dx}{\sqrt{x^2-8x+32}}$
9. $\frac{dx}{\sqrt{3x^2-4x+1}}$
10. $\int \frac{3x+10}{\sqrt{x^2+2x+5}}$

UNIT 7

FURTHER TECHNIQUES OF INTEGRATION II

TABLE OF CONTENTS

1.0 INTRODUCTION

As have been mentioned in previous section, integration is a process that involves anti-differentiation. You start by making a guess and determine whether the differentiation of your guess can give you the function you want to integrate. The techniques you have studied so far are all trying to narrow your guess to the exact solution. Therefore in this unit you will study rational functions can be integrated by first resolving the rational function into partial fractions. The method of resolving rational fraction into partial fraction before integration is called integration by method of partial fractions. Also there are functions that are formed as a product of exponential functions and trigonometric function. Finding, integrals of such product functions can only be possible by making use of the product rule studied in the first course in calculus i.e. calculus I. This type of integration by which the product rule is applied is known as integration by parts. All these techniques are studied in order to make it easier for you to evaluate integrals of function without making a wide guess. So endeavour to practice the examples given here and in the previous units.

2.0 OBJECTIVES

After studying this unit you should be able to:

i. integrate certain types of rational functions by method of partial fractions.

ii. evaluate integrals of product functions by method of integration by parts.

3.0 INTEGRATOINS BY PARTIAL

In the course in algebra MTH 111 you studied how to split a rational function into a sum of fractions with simpler denomination. You were told that the process of doing this is called the method of partial fractions. It is advisable that you review this method of partial fraction in the materials given to you for the course in algebra. You could recall that

$$\frac{x+7}{6+x-x^2} = \frac{x-7}{(3-x)(2+x)} = \frac{2}{3-x} + \frac{1}{2+x}$$

therefore if you want to integrate a rational function of this type, $\int \frac{f(x)}{g(x)}\,dx$

There are two things you will check before you decide to use the method of partial fractions. These are as follows:

1. The degree of f(x) should be less than the degree of g(x). If this is not the case, you must perform a long division, then resolve the remainder into partial fraction.

2. The factors of g9x) should be known by you.

Example: Find $\int \frac{x+7}{6+x-x^2}\,dx$

Solution: The degree of x + 7 is lower than that of $6+x-x^2$.

The factors of $6+x-x^2$ are (3-x)(2+x)

$$\text{Therefore, } \int \frac{x+7}{6+x-x^2}\,dx = \int \frac{2}{3-x}\,dx + \int \frac{dx}{2+x}$$

$$= -2\ln/3-x/+\ln/2+x/+C.$$

Example: Evaluate $\int \frac{4x^2-24x+11}{(x+2)(x-3)^2}$

Solution: The degree of $4x^2 - 24x + 11$ is less than that of $(x+2)(x-3)^2$

$$\frac{4x^2-24x+11}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

i.e. $4x^2 - 24x + 11 = (A + B)x^2 + (C - 6A - B)x + 9A - 6B + 2C$

Equating coefficient and solving the resolutions simultaneous equations yields

$$\frac{4x^2 - 24x + 11}{(x+25\ x - 3)^2} = \frac{3}{x+2} + \frac{1}{3-3} - \frac{5}{(x-3)^2}$$

therefore $\int \frac{4x^2 - 24x + 11}{(x+2)(x-3)^2} dx = \int \frac{3}{x+2} du + \int \frac{dx}{x-3} - \int \frac{5dx}{(x-3)^2}$

$$= 3\ln/x+2/+\ln/x-3/+\frac{5}{(x-2)} + C$$

Example: Find $\int \frac{6x + 1}{3x^3+12x^2-2x-3}$

Solution: $3x^3 + 12x^2 - 2x - 3 = (4x^2 -1)(2x + 3)$.

Then $\frac{6x + 1}{(4x^2-1)(2x+3)} = \frac{A}{2x -1} + \frac{B}{2x+1} + \frac{C}{2x+3}$.

$6x + x = A (2x+1)(2x+3) + B(2x-1)(2x +3) + C(2x-1)(2x+1)$

By equations coefficients and solving the resulting simultaneous equation you get

$$\frac{6x + 1}{(4x^2-1)92x+3)} = \frac{1}{2(2x-1)} - \frac{1}{2(2x+1)} - \frac{1}{2x + 3}$$

therefore: $\int \frac{6x + 1}{3x^3+12x^2-2x-3} dx = ½ \int \frac{dx}{2x-1} + ½ \int \frac{dx}{2x+1} - \int \frac{dx}{2x+3}$

$$= ¼ \ln/2x-1/+1/ ¼ \ln/2x+1/- ½ \ln/2x+3/ + C.$$

Example: Find $\int \frac{3x^2}{1 + x^3} dx$

Solution: $\frac{3x^2}{1 + x^3} = \frac{A}{1 + x} + \frac{Bx + C}{1 - x + x^2}$

Therefore $3x^2 = A(1 - x + x^2) + (Bx + C) (1 + x)$

Equations coefficient solving the resulting simultaneous equations yields

$$\frac{3x^2}{1+x^3} = \frac{1}{1+x} + \frac{2x - 1}{1-x+x^2}$$

therefore: $\int \frac{3x^2}{1+x^3} dx = \int \frac{1}{1+x} dx + \int \frac{2x - 1}{1-x+x^2} dx$

Example: Find $\int \frac{x + 3}{}$

$$4x^3+4x^2-7x+2$$

Solution: $4x^3+4x^2 - 7x + 2 = (2x - 1)(2x - 1)(x + 2)$

then $\dfrac{x + 3}{(2x-1)^2(x+2)} = \dfrac{A}{2x-1} + \dfrac{B}{(2x-1)^2} + \dfrac{C}{x + 2}$

hence $x + 3 = A(2x-1)(x+2) + B(x+2) + C(2x - 1)^2$

Equating coefficients and solving the resulting simultaneous equations yields

$$\frac{x + 3}{(2x-1)^2(x+7)} = \frac{-2}{25(2x - 1)} + \frac{7}{5(2x - 1)} + \frac{1}{25(x + 2)}$$

therefore: $\displaystyle\int \frac{x + 3}{4x^3+4x^{2-7x+2}}\,dx = \frac{-2}{25}\int\frac{dx}{2x-1} + \frac{7}{5}\int\frac{dx}{(2x-1)^2} + \frac{1}{25}\int\frac{dx}{x+2}$

$$= \frac{-1}{25}\ln/2x-1/ \frac{-7}{10(2x-1)} + \frac{1}{25}\ln/x+2/+C$$

Exercises: Find the following integrals

(1) $\displaystyle\int \frac{dx}{(x+1)(x+2)}c$

(2) $\displaystyle\int \frac{3x - 1}{(2+x)(x-3)}\,dx$

(3) $\displaystyle\int \frac{x - 5}{(x-1)(x+4)}\,dx$

(4) $\displaystyle\int \frac{2x^2 + x + 6}{4x^2 - 4x - 3}\,dx$

(5) $\displaystyle\int \frac{x - 3}{x^2+2x-8}\,dx$

(6) $\displaystyle\int \frac{5}{1+3x+ 2x^2}\,dx$

(7) $\displaystyle\int \frac{x^2}{2x^2 - 3x - 2}\,dx$

(8) $\displaystyle\int \frac{x^3 + x + 1}{x^2 - 1}\,dx$

(9) $\displaystyle\int \frac{2 - x^2}{(2x+1)^3(x-1)}\,dx$

(10) $\displaystyle\int \frac{x^3}{(x^2+x+4)(x^2+1)}\,dx$

Ans:

(1) $\ln(x+1) - \ln/x +2/ + C$

(2) $7/5 \ln/x+2/ + 8/5 \ln/x-1/ + C$

(3) $9/5 \ln/x+4/ - 4/5\ln/x - 1/ + C$

(4) $\frac{x}{2} - \frac{3}{4} \ln/2x + 1/ + \frac{3}{2} \ln 2x - 3/ + C$

(5) $\frac{7}{6} \ln/x + 4/ - \frac{1}{6} \ln/x - 2/ + C$

(6) $5\ln(2x + 1) - 5\ln/x + 1/ + C$

(7) $\frac{x}{2} - \frac{1}{20} \ln(2x + 1) + \frac{4}{5} \ln/x - 2/ + C$

(8) $\frac{1}{2} x^2 + \underline{3} \ln/x-1/+ \frac{1}{2} \ln/x + 1/ + C$

(9) $\frac{7}{24(2x+1)^2} + \frac{13}{36(2x+1)} - \frac{1}{27} \ln/2x + 1/+ \frac{1}{27} \ln/x - 1/ + C$

(10) $\frac{13}{20} \ln/x^2 + x + 4/ - \frac{\sqrt{15}}{30} \arctan\left(\frac{2x + 1}{\sqrt{15}}\right) \frac{-3}{20} \ln/x^2 + 1/$

$- \frac{1}{10} \arctan x$

3.1 INTEGRATION BY PARTS

The method of integration by parts owns its origin to the differential of a product. That is d(uv) = u dv + v du – (i)
or udv = u dv – v du - (ii)

integrating equation (ii) you get $\int u\,dv = \int d(uv) - \int v du$
$\int u\,dv = uv - \int v\,du. + C$ – III

Equation III above expresses one integral $\int u\,dv$, in terms of a second integral $\int u\,dv$. The idea behind this method is that, if by appropriate choice U and dv, the second integral is simpler than the first, you may be able to evaluate it quite simply and as such arrive at the solution.

Example: Find $\int x \sin x\,dx$

Solution: let $\int x \sin x\,dx = -\int x\,d(\cos x)$

then using the formula for integration by part given in equation III above you have that $\Longrightarrow$ v = cos x
therefore $\int x \sin x\,dx$

$= \int u\,dv = uv - \int v\,du$

$= -\cos x\,(x) + \int \cos x\,dx$

$= -x \cos x + \sin x + C$

Example: $\int xe^{-x}\, dx$.

Solution: Use integration by parts with u = x, du = dx, dv = e^{-x} dx, v =$-e^{-x}$

Therefore $\int xc^{-x}\, dx = -xe^{-x} - \int -e^{-x}\, dx$
$= -xe^{-x} - c^{-x} + c$

In above example, it is possible to choose u and v differently
i.e. $\int xe^{-x}\, dx = \int udv$
$u = e^{-x}$, $du = -e^{-x}\, dx$ $dv = x\, dx$ $v = \frac{x^2}{2}$
then integration by parts you get

$$\int xe^{-x}\, dx = \frac{x^2e^{-x}}{2} - \int -^{-x} \frac{x^2}{2}\, dx$$

The above is true but the resulting integral on the right is harder than the given one on the left. Therefore, you should be cautious when factoring the integrand into u and dv. With more examples, you get use to this technique of integration by parts.

Examples: Find $\int \ln x\, dx$

Solution: $u = \ln x$, $dv = dx$, $du = \frac{1}{x}\, dx$

therefore $\int \ln dx = x\ln x - \int x.\frac{1}{x}\, dx$.
$= x\ln x - x. + c.$

Example: Find $\int \text{arc} \cos x\, dx$

Solution: $u = \text{arc} \cos x$ $du = \frac{-\,dx}{\sqrt{1-x^2}}$
$dv = dx$, $v = x$.

therefore

$$\int \text{arc} \cos x\, dx = uv - \int vdu = x \text{ arc} \cos x - \int \frac{-x\, dx}{\sqrt{1-x^2}}$$

but $\int \frac{x\, dx}{\sqrt{1-x^2}} = -\int \frac{u\, du}{y}$ where $y^2 = 1 - x^2$

$$-\int dy = -y + c = \frac{-ydu}{\sqrt{1-x^2} + C} = + x\, dx$$

therefore: $\int \arccos x \, dx = x \arccos x - \sqrt{1-x^2} + C$

Example: Find $\int \frac{xe^x \, dx}{(x+1)^2}$

Solution: After several attempts it is found that the following factors for the integrand will work.

$$\text{Let } u = xe^x \quad du = \frac{1}{(x+1)^2} dx$$

$$du = (xe^x + e^{x)\,dx\,=}\, e^x (x+1)\, dx$$

$$v = -\frac{1}{x+1}$$

therefore: $\int \frac{xe^x}{(x+1)^2} dx = uv - \int v du = \frac{xe^x}{(x+1)} - \int \frac{e^x(x+1)}{-(x+1)}$

$$= \frac{-xe^{-x}}{x+1} + \int e^x \, dx$$

$$= -\frac{xe^x}{x+1} + e^x + C$$

$$= \frac{e^x}{x+1} + C$$

3.1.1 REPEATED INTEGRATION BY PARTS

Some integration may require that you apply the method of integration by parts two or more times.

Example: Find $\int x^2 e^x \, dx$.

Solution: Applying integration by parts you get:

$u = x^2, dv = e^x \, dx$

$du = 2x \, dx, \; v = e^x$

therefore

$$\int x^2 e^x \, dx = \int udv = uv - \int v \, du$$

$$= x^2e^x - \int 2xe^x dx$$

To find $\int 2x \, e^x \, dx$, you apply integration by parts again. By letting

$u = 2x \quad dv = e^x\, dx$
$du = 2\, dx \quad v = e^x$

then $\int 2x\, e^x = \int u dv = uv - \int v\, du$
$= 2x\, e^x - \int 2e^x\, dx$
$= 2x\, e^x - 2e^x + C$

hence $\int x^2 e^x\, dx = x^2 e^x - 2x\, e^x + 2e^x + C$

Example: $\int x(\ln x)^2\, dx$

Solution: let $u = (\ln x)^2 \quad du = \frac{2\ln x\, dx}{X}$

$dv = x\, dx \quad v = \frac{x^2}{2}$

thus $\int x(\ln x)^2\, dx = \int u dv = vu - \int v du$

$= \frac{x^2}{2} (\ln x)^2 - \int \frac{x^2}{2} . 2\frac{\ln x}{x}\, dx$

$= \frac{(x\ln x)^2}{2} - \int x\ln x\, dx$

To evaluate $\int x \ln x\, dx$, you apply integration by parts the second time.

i.e. $u = \ln x \quad du = \frac{dx}{x} \quad dv = x\, dx \quad v = \frac{x^2}{2}$

therefore $\int x\ln x = \frac{x^2}{2} . \ln x - \int \frac{dx}{x} . \frac{x^2}{2}$
$= \frac{x^2 \ln x}{2} - \int \frac{x}{2}\, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

thus

$\int x(\ln x)^2 = \frac{(x\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C$

Example: Find $\int x^3 e^x\, dx$

Solution: You will apply integration by parts three times to get the solution.

$\int x^3 e^{-x}\, dx$

$dv = e^{-x}\,dx$, $u = x^3$ $du = 3x^2dv$ $v = e^x$.

$\int x^3e^{-x}dx = uv - \int vdu = -x^3e^{-x} - \int -e^{-x}\, 3x^2\, dx$

$3\int e^{-x}\, x^2\, dx = 3[uv - \int vdu]$

$u = x^2$ $dv = e^{-x}\, dx$, $du = 2x\, dx$, $v = -e^{-x}$

$3\int e^{-x}\, x^2\, dx = 3 - x^2\, e^{-x} - 3\int -e^{-x}\, 2x\, dx$.

$+ 6\int e^{-x}\, x\, dx = 6[uv - \int vdu]$

$u = x$, $du = dx$, $dv = e^{-x}\, dx$, $v = -e^{-x}$

$6\int e^{-x}\, x\, dx = -6x\, e^{-x} - 6\int -e^{-x}\, dx$
$= 6xe^{-x} + 6e^{-x} + C$

therefore: $\int x^3e^{-x}\, dx = -x^3\, e^{-x} - 3x^2\, e^{-x} - 6x\, e^{-x} - 6\, e^{-x} + C$
$= e^{-x}\,(-x^3 - 3x^2 - 6x - 6) + C$

Example: Find $\int e^x \sin x\, dx$

Solution: let $u = e^x$, $du = e^x\, dx$ $du = \sin x\, dx$ $v = \cos x$

Let
$I = \int e^x \sin x\, dx$

therefore $I = uv - \int v\, du$
$= -e^x \cos x - \int - \cos x\, e^x\, dx$.
$\int \cos x\, e^x\, dx$ integrate by parts again by letting $u = e^x$ $du = e^x\, dx$, $dv=\cos x\, dx$.
$v = \sin x$
then $\int \cos x\, e^x\, dx = e^x \sin x - \int \sin x\, e^x\, dx$.

Therefore $I = -e^x \cos x + e^x \sin x - I$ (since $I = \int \sin x\, e^2\, dx$)
$\Rightarrow$ $2I = -e^x \cos x + e^x \sin x$.

therefore $I = \dfrac{e^x(\sin x - \cos x)}{2}$

$\Rightarrow$ $\int \sin x\, e^x\, dx = \dfrac{e^x(\sin x - \cos x)}{2} + C$

Exercise: Evaluate the following integrals

(1) $\int xe^x\, dx$ (2) $\int x \cos x\, dx$

(3) $\int x^2\, e^{2x}\, dx$ (4) $\int x^2\, e^{-x}\, dx$

(5) $\int xe^{2x}\, dx$ (6) $\int \ln (x^2 + 1)\, dx$

(7) $\int x \sec^2 x \, dx$ (8) $\int x^3 e^{x^2} dx$

(9) $\int x^2 \cos x \, dx$ (10) $\int e^{2x} \sin x \, dx$

(11) $\int x \tan^2 x \, dx$ (12) $\int e^{2x} \cos x \, dx$

Ans

(1) $xe^x - e^x + C$

(2) $\cos x + x \sin x + C$

(3) $^1e^{2x} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) + C$

(4) $-e^{-x} (-x^2 + 2x - 2) + C$

(5) $e^{2x} \left(\frac{x}{2} - \frac{1}{4}\right) + C$

(6) $x\ln (x^2 + 1) - 2x + 2 \arctan x + C$

(7) $x \tan x + \ln (\cos x) + C$

(8) $-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x + C$

(9) $x^2 \sin x - 2 \sin x + 2x \cos x + C$

(10) $e^2x \left(-\frac{\cos x}{5} + \frac{2}{5} \sin x\right) + C$

(11) $x \tan x - \frac{x^2}{2} - \frac{\ln (1 + \tan^2 x)}{2} + C$

(12) $e^{2x} \left(\frac{2}{5} \cos x + \frac{1}{5} \sin x\right) + C$

4.0 CONCLUSION

You have studied two techniques of integration. The method of partial fraction requires that you factorize the denominator so that you could have simpler factors which in turn will be easier to integrate. Also you used the technique of integration to integrate product of functions that are somehow difficult to integrate. Breaking up the integration into parts by applying the product rule for differentiation yields simpler integrands that you are already familiar with in previous units of this course. In the next unit you will study a technique which is very similar to the technique of integration by parts. In this unit various solved examples have been provided for you. This is because understanding the examples will enable you to know at glance if a particular integration should be carried out by any of the techniques studie in this unit. Doing all the exercises provided in this unit will also sharpen your skills in the use of the techniques studied in the unit.

5.0 SUMMARY

In this unit you have studied

(i) the technique of integration by partial fraction.

i.e. $\int \frac{f(x)}{g(x)} dx = \int \frac{A_1}{g_1(x)} dx + \int \frac{A_2}{g_2(x)} dx + \int \frac{An}{g_n(x)} dx$

where $g(x) = g_1(x)\, g_2(x) \ldots g_n(x)$ and the integrals on the left and are simpler than the given integral on the left.

(ii) how to integrate product of function such as xe^x, $e^x \sin x$ x lnx etc by the technique of integration by parts.

i.e. $\int u\, dv = uv - \int v\, du$

where the integral on the left is simpler than the given integral on the left.

(iii) how to apply the method of integration by parts two or more times.

6.0 TUTOR MARKED ASSIGNMENT

Find the following integrals.

1. $\int \frac{1}{x^2 - 4} dx$

(2) $\int \frac{5x - 3}{(x+1)(x-3)} dx$

3. $\int \frac{-2x + 4}{(x^2+1)(x-1)^2}$

(4) $\int \frac{2x^2 + 3}{x(x-1)^2} dx$

(5) $\int \frac{x^2 + 1}{x^3 - 4x^2 + x + 6} dx$

(6) $\int \arctan x\, dx$

(7) $\int x^3 e^x\, dx$

(8) $\int x^2 \cos ax\, dx$

(9) $\int \sin(\ln x)\, dx$

(10) $\int_1^2 x \sin ax\, dx$

UNIT 8

FURTHER TECHNIQUES OF INTEGRATION III

TABLE OF CONTENTS

1.0 INTRODUCTION

In this unit you shall study two additional techniques for integration. The previous unit ended with the technique of integration by parts. In this unit you shall extend the method of integration by parts to derive what is known as reduction formulas for certain categories of product functions. The reduction is derived by applying the method of integration by parts repeatedly until the power of one of the product functions is reduced to 1 or 0. the second technique that will be studied in this unit involves using an appropriate substitution which makes it possible to integrate all rational expression of sin x and cos x, then the substitution 2 arc tan u = x transforms the integral $\int f(x)$ into the integral of a rational function of u, which can be evaluated by techniques studied in previous units. The third technique that will be studied in this unit involves using appropriate substitution for rational expression containing some radicals such as $\sqrt{x}$, $x^{3/4}$, $\sqrt{1-e^x}$ etc.

2.0 OBJECTIVES

After studying this unit you should be able to correctly

(1) derive reduction formulas for integrals such as
(i) $\int \cos^n x\, dx$, (ii) $\int \sin^n x\, dx$, (iii) $\int e^{ax} \cos bx\, dx$, etc

(2) evaluate integrals of rational functions of sin x and cos x using the substitution $u = \tan x/2$.

(3) evaluate integrals rational functions involving radicals and fractional powers of x.

(4) evaluate integrals of product function of trigonometric ratios involving cos bu cos ax sin sub u cos ax and au bus u ax.

3.0 REDUCTION FORMULAS

Repeated application of integration by parts could reduce the power of a function from n, say + 1 or 0. Thus a formula could emerge from the above application that can be used for evaluating integrals which are similar.

Given the integral $\int x^2 e^x\, dx$ which you are quite familiar with in unit 7. The integral above requires two integration by parts. Each integration lowers the power of x by one until x disappears. In a similar way the integral $\int x^3 e^x\, dx$ requires three integration by parts and the integral $\iint x^4 e^x\, dx$ require four integration by parts. This process can continue for any power of x say n. Thus to evaluate the integral $\int x^n e^x\, dx$ requires ^{n} integration by parts.

Example: Obtain a reduction formula for the integral $J_n = \int x^n e^x\, dx$.

Solution: Integrate by parts setting $u = x^n$, $du = nx^{n-1}\, dx$, $dv = e^x$ $v = e^x$.

$$\text{Then } J_n = \int x^n e^x\, dx = uv - \int v\, du$$

$$\text{therefore } J_n = x^n e^x - \int nx^{n-1} e^x\, dx$$

$$= x^n e^x - n \int_{n-1}$$

Thus the reduction formula is given as $J_n = x^n e^x - n\int_{n-1}$ ---- (A)

Example: Find $\int x^{6x} e_x\, dx$

Solution: Use the reduction formula of equation (A) to evaluate J_4. By the reduction formula in equation (A) n = 6, that is $J_6 = x^6 e^x - 6\int_5$
By the reduction formula with n = 5 $J_5 = x^5 e^x - 5 J_4$
By the reduction formula with n = 4 $J_4 = x^4 e^x - 4 J_3$
Thus $J_6 = x^6 e^x - 6 (x^5 e^x - 5 J_4)$
$= x^6 e^x - 6x^5 e^x + 30 \int_4$

But by repeated use of the reduction formula

$$\begin{aligned} J_4 &= x^4 e^x - 4\int_3 \\ &= x^4 e^x - 4(x^3 e^x - 3\int_2) \\ &= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2\int_1) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 (xe^x - \int_o) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C \\ \text{thus } J_4 &= e^x (x^4 - 4x^n + 12x^2 - 24x + 24) + C \end{aligned}$$

$$\text{therefore } J_6 = e^x (x^6 - 6x^5 + 30 (x^4 - 4x^3 + 12x^2 - 24x + 24) + C$$

$= e^x (x^6 – 6x^5 + 30x^4 – 120x^3 + 360x^2 – 720x + 720) + C$

Example: Obtain a reduction formula for $\int \sin^n x \, dx$.

Solution: You can write $J_n = \int \sin^n x \, dx = \int (\sin^{n-1}x)(\sin x) \, dx$
and integrate by parts with

$u = (\sin x)^{n-1}$ $du = (n-1)(\sin x)^{n-2} \cos x \, dx$
$dv = \sin x \, dx$ $v = -\cos x$

therefore $\int \sin^n x \, dx = (\sin x)^{n-1} (-\cos x) + \int (n-1) \sin x^{n-2} \cos^2 dx$

therefore $= J_n \sin^n x \, dx = (\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x (1- \sin^2 x) \, dx$

$= (\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x \, dx - \int \sin^n x \, dx$
$J_n \quad = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2} – (n-1) J_n$

collecting like terms you get

$(n-1) \int_n + \int_n = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2}$

$n \int_n = (\sin x)^{n-1} \cos x + (n-1) \int_{n-2}$
dividing through by n you obtain

$$Jn = \frac{(\sin x)^{n-1} \cos x}{n} + \frac{n-1}{n} \int_{n-2}$$

In the above the reduction formula lowers the power of sin x by two. Therefore, repeated application will reduce $\int_n$ to $\int_o$ or $\int_1$ accordingly as n is even or odd i.e.

$J_n = \int \sin x \, dx = \cos x + C$
$J_o = \int dx = x + C$

Example: $\int_0^{\pi/2} (\sin x)^6 \, dx$

Solution: set $J_n = \int^{\Pi/2} (\sin x)^6 \, dx$

Using the reduction formula of the last example, you get

Let $J_6 = \frac{(\sin x)^5 \cos x}{6} + \frac{5}{6} J_4$
for brevity you write

$J_6 = I, + \frac{5}{4}[\frac{\sin^3 x \cos x}{4} + \frac{3}{4} J_2]$

$= I + \frac{5}{6}[I_2 + \frac{3}{4}[I_3 + ½ J_o]$

$I_1 = \frac{(\sin x)^5 \cos x}{6} /_0^{II/2} = \frac{(\sin^{II/2})^5 \cos^{II/2}}{6} - \frac{(\sin o)^5 \cos o}{6} = 0$

$I_2 = 0, \quad I_3 = 0$

hence

$J_6 = \frac{5}{6}(¾)(1/2\ J_o) = \frac{5}{6}(¾)(½)(x)/_0^{II/2}$

$= \frac{5}{6} . \frac{3}{4} . ½ . \frac{II}{2} . = \frac{5_{II}}{32}$

Exercises: Find (1) $\int \sin^5 x\, dx$, (2) $\int_0^{II} \sin^7 x\, dx$

(3) $\int x^5 e^x\, dx$ (4) $\int_0^1 x^7 e^x\, dx$

(5) $\int \cos^3 x\, dx$

Ans:

(1) $e^x (x^5 - 5x^4 + 20 x^3 - 60 x^4 + 120 x - 120)$

(1) $-\frac{1}{15}\cos x (3 \sin^4 x + 4 \sin^2 x + 8) + C$

(2) $\frac{32}{35}$

(3) $e^x (x^5 - 5x^4 + 20x^3 - 60x^4 + 120x - 120) + C$

(4) $5040 - 1854e^1$

(5) $\frac{1}{3}\sin x (\cos^2 x + 2) + C$

Example: Obtain a reduction formula for $\int e^{ax} \cos bx\, dx$.

Solution: Let $U = e^{ax}$ and $dv = \cos bx\, dx$

Then $du = ae^{ax}\, dx$ and $v = \frac{1}{B}\sin bx$

Hence $\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{} - \int \frac{1}{} \sin bx . ae^{ax}\, dx$

B b

(A) therefore $\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b}\int \sin bx\, e^{ax}\, dx$

The integral on the right is like the first one except that it has sin bx instead of cos bx. You will apply integration by part again by letting $u = e^{ax}$ and $dv = \sin bx\, dx$. then $du = ae^{ax}\, dx$ and $v = -\frac{1}{b}\cos bx$

hence

(B) $\int \sin bx\, e^{ax}\, dx = -\frac{e^{ax} \cos bx}{b} - \int -\frac{a}{b} e^{ax} \cos bx\, dx$

$$= -\frac{e^{ax} \cos bx}{} + \frac{a}{}\int e^{ax} \cos bx\, dx$$

therefore: substituting equation (B) into equation (A) you get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b}\left(-\frac{e^{ax} \cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx\, dx\right)$$

$$= \frac{e^{ax} \sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2}\int e^{ax} \cos bx\, dx$$

collecting like terms you get

$$\int e^{ax} \cos bx\, dx \left(1 + \frac{a^2}{b^2}\right) = \frac{e^{ax}}{b}\left(\sin bx + \frac{a}{b} \cos bx\right)$$

therefore $\int e^{ax} \cos bx\, dx = e^{ax}\left(\frac{\sin bx}{b} + \frac{a}{b}\cos bx\right)\left(\frac{b^2}{a^2+b^2}\right)$

$$= e^{ax}\left(\frac{\sin bx + a \text{ cps } bx}{a^2 + b^2}\right) + C$$

Example: Find $\int e^{2x} \cos 3x\, dx$

Solution: Using the above reduction formula you have that a = 2 and b = 3 then

$$\int e^{2x} \cos 3x\, dx = e^{2x}\frac{(3 \sin 3x + 2 \cos 3x)}{4 + 9} + C$$

$$= \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$$

Example: Find $\int e^{x/2} \cos 2/3\, x\, dx$.

Solution: let a = ½ b = 2/3 then by the reduction formula

$$\int e^{ax} \cos bx \, dx = e^{ax} \frac{(b \sin bx + a \cos bx)}{a^2 + b^2} + C$$

you have that

$$\int e^{x/2} \cos 2/3 \, x \, dx = e^{x/2} \frac{(2/3 \, x + ½ \cos 2/3 \, x)}{¼ \; + \; 4/9}$$

$$= \frac{36}{25} e^{x/2} (2/3 \sin 2/3x + ½ \cos 2/3x) + C$$

3.1 RATIONAL EXPRESSIONS IN SIN X AND COS X

There are certain class of trigonometric functions that techniques studied in the previous might not be able to be used to integrate them specifically, rational functions of sinx and cos x. An appropriate substitution of u = tan x/ 2 might reduce the problem of integrating such class of rational functions of sinx and cox to a problem of integrating a rational function of u. This in turn can be integrated by the method of partial fraction studied in unit 7.

Example: If f(x) is a rational expression in sinx and cosx, then the substitution u = tan x/2 transforms the integral ∫ f(x) dx into the integral of a rational function of u.

Solution: A typical way to explain the above is to start by expressing cosx and sinx in terms of u.

i.e. $\cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{\sec^2(\frac{x}{2})} -$

$$= \frac{2}{1 + \tan^2(\frac{x}{2})} - 1 = \frac{2}{1 + u} -$$

$$= \frac{1 - u^2}{1 + u^2}$$

therefore: $\cos x = \frac{1 - u^2}{1 + u^2}$, $u = \tan \frac{x}{2}$

and $\sin x = 2 \sin \frac{x}{} \cos \frac{x}{} = 2 \frac{\sin x/2}{} . \cos^2 \frac{x}{}$

2 2 cos(x/2) 2

$$= 2\tan(x/2)\,.\,\frac{1}{\sec^2 \frac{x}{2}} -$$

$$= \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

$$= \frac{2u}{1+u^2}.$$

therefore $\sin x = \frac{2u}{1+u^2}$, $u = \tan x/2$

$x = 2$ ac tan u

$dx = \frac{2\,du}{1+u^2}.$

Use the above tools to evaluate

(1) $\int \sec x\, dx$

Solution: $\int \sec x\, dx = \int \frac{dx}{\cos x}$

from above $\cos x = \frac{1-u^2}{1+u^2}$ $dx = \frac{2\,du}{1+u^{22}}$

therefore: $\int \frac{dx}{\cos x} = \int \frac{2\,du}{1+u^2}.\frac{1+u^2}{1-u^2} = 2\int \frac{du}{1-u^2}$

by partial fractions you arrive at $2\int \frac{du}{1-u^2} = \int \frac{A\,du}{1-u} + \int \frac{B\,du}{1+u}$

$$= \int \frac{du}{1-u} + \int \frac{du}{1+u}$$

$$= \ln/1-u/+\ln/1+u/+C$$

$$= \ln/\frac{1+u}{1-u}/ + C$$

but $u = \tan x/2$ thus

$$\int \frac{dx}{} = \ln/\frac{1+\tan x/2}{1-\text{tax } x/2}/ + C$$

Example: $\int \frac{dx}{1 + 2\cos x}$

Solution: let $\cos x = \frac{1 - u^2}{1 + u^2}$

then $1 + 2\cos x = 1 + 2\left(\frac{1 - u^2}{1 + u^2}\right) = \frac{1 + u^2 + 2 - 2u^2}{1 + u^2}$

$$= \frac{3 - u^2}{1 + u^2}$$

$dx = \frac{2\,du}{1 + u^2}$.

therefore: $\int \frac{dx}{1 + 2\cos x} = \int \frac{2\,du}{1 + u^2} \cdot \frac{1 + u^2}{3 - u2} = \int \frac{2\,du}{3 - u^2}$.

therefore: $\int \frac{2dv}{3 - u^2} = 2\int \frac{du}{u^2 - 3} = \frac{1}{\sqrt{3}}\left(\int \frac{du}{u + \sqrt{3}} - \int \frac{du}{u - \sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}(\ln/u + \sqrt{3}/ - \ln/u - \sqrt{3}/) + C$$

$$\sqrt{3}\ln/\frac{\tan^{2/2} + \sqrt{3}}{\tan^{2/2} - \sqrt{3}}/ + C$$

Example: Find $\int \frac{dx}{5\sec x - 3}$.

Solution: set $\cos x = \frac{1 - u^2}{1 + u^2}$, $dx = \frac{du}{1 + u^2}$

where $u = \tan x/2$

$$5\sec x - 3 = \frac{5}{\cos x} - 3 = 5\frac{(1 - u\,u^2)}{1 + u^2} - 3$$

$$= \frac{5(1 - u^2) - 3(1 + u^2)}{1 + u^2} = \frac{2 - 8u^2}{1 + u^2}$$

therefore $\int \frac{d\,x}{5\sec - 3} = \int \frac{2du}{2 - 8u^2} = \int \frac{du}{1 - 4u^2}$

$$= \tfrac{1}{4}\left[\int \frac{du}{1-2u} + \int \frac{du}{1+2u}\right]$$

$$= \tfrac{1}{4}[-\ln/(2u - 1) + \ln(2u + 1)/$$

$= \frac{1}{4} \ln\left|\frac{2u+1}{2u-1}\right| + C$

$= \frac{1}{4} \ln\left|\frac{2(\tan^{x/2})+1}{2(\tan^{x/2})-1}\right| + C$

Example: Find $\int \frac{1}{5 + 4\cos x} dx$

Solution: set $\cos x = \frac{1-u^2}{1+u^2}$ $dx = \frac{2\,dx}{1+u^2}$

therefore $5 + 4\cos x = 5 + \frac{4(1-u^2)}{1+u^2}$

$= \frac{5(1+u^2) + 4(1-u^2)}{1+u^2}$

$= \frac{9+u^2}{1+u^2}$

$\int \frac{dx}{5+4\cos x} = \int \frac{2\,du}{1+u^2} \cdot \frac{1+u^2}{9+u^2}$

$= \int \frac{2\,du}{9+u^2} = 2\int \frac{du}{9+u^2}$

$= 2\int \frac{du}{3^2+u^2} = = \frac{2}{3} \arctan \frac{u}{3}$

therefore $\int \frac{dx}{5+4\cos x} = \frac{2}{3} \arctan \frac{u}{3} = \frac{2}{3} \arctan \frac{1}{3}(\tan \frac{x}{2})$

Exercises: Evaluate the following integrals

(1) $\int_0^{\Pi/2} \frac{dx}{4+\cos x}$

(2) $\int \frac{\sin x}{3-\sin x} dx$

(3) $\int \frac{1+\sin x}{1+\cos x} dx$

(4) $\int \frac{1-\sin x}{1+\cos x} dx$

(5) $\int \frac{1+\sin x}{1+\cos x} dx$

Ans:

(1) $\frac{2}{15}\sqrt{15}$ arc tan $(\frac{1}{5}\tan(\frac{x}{2})\sqrt{15}) + C$

(2) $\frac{1}{3}\ln(3 + 3(\tan^2\frac{x}{2}) - 2\tan\frac{x}{2}) + \frac{\sqrt{2}}{6}$ arc tan $(\frac{\sqrt{2}}{8}(6\tan\frac{x}{2} - 2)$

(3) $\frac{2}{7}\sqrt{7}$ arc tan $(\frac{7}{7}\tan\frac{x}{2})$

(4) $\tan\frac{x}{2} - \ln(1 + \tan^2\frac{x}{2})$

(5) $\tan\frac{x}{2} + \ln(1 + \tan^2\frac{x}{2})$

3.2 FURTHER SUBSTITUTIONS

You are quite familiar with the following trigonometric identities;

(1) sin px sin tx = ½ [cos (p - t) x - cos (p + t) x]

(2) sin px cos t x = ½ [sin (p – t) x + sin (p + t) x]

(3) cos px cos t x = ½ [cos (p – t) x + cos (p + n) x]

using the above evaluate

(i) $\int \sin 3x \cos 7x\, dx = \frac{1}{2}\int \sin(3-7)x + \sin(3+7)x\, dx$
$= \frac{1}{2}\int(\sin 10x - \sin 4x)\, dx$
$= -\frac{\cos 10x}{20} + \frac{\cos 4x}{8} + C$

(ii) $\int \cos 2x \cos 3x\, dx = \frac{1}{2}\int \cos(3-2)x + \cos(2+3)x\, dx$
$= \frac{1}{2}\int(\cos x + \cos 5x)\, dx$
$= \frac{\sin x}{2} + \frac{\sin 5x}{5} + C$

(iv) $\int \sin 7x \sin x\, dx = \frac{1}{2}\int \cos(7-1)u - \cos(t+1)x$
$= \frac{1}{2}\int(\cos 6x - \cos 8x)$

$\frac{1}{2}[\frac{\sin 6x}{} - \frac{\sin 8x}{}] = \frac{\sin 6x}{} - \frac{\sin 16x}{} + C$

6 8 12 2

Exercises: Evaluate the following integrals:

(i) $\int_{\Pi}^{\Pi} \sin 2x \sin 5x \, dx$ (ii) $\int \cos 5x \cos 6x \, dx$

(iii) $\int_{o}^{\Pi} \cos 5x \sin 6x \, dx$

3.3 OTHER RATIONALISING SUBSTITUTIONS

You will study how to integrate rational functions f(x) involving fractional powers of x. This you will carry out by using the substitution $x = u^n$.

Example: Find $\int \frac{dx}{1 + \sqrt{x}}$

Solution: set $u^2 = x$ $2u\, du = dx$ where $u = \sqrt{x}$ then

$$\int \frac{dx}{1 + \sqrt{x}} = \int \frac{2\, u\, du}{1 + u} - 2\int \frac{u\, du}{1 + u}$$

$$= \quad 2 \int (1 - \frac{1}{1+u}) \, du$$

$$= \quad 2(u - \ln 11 + u/ + C$$

$$= \quad 2u - 2\ln/1 + u/ + C$$

$$= \quad 2\sqrt{x} - 2\ln/1 + \sqrt{x}/ + C$$

Example: Find $\int \frac{d\ x}{1 + {}^4\sqrt{x}}$

Solution: Set $u^4 = x$ $4u^3\, du = dx$

Then $\int \frac{dx}{1 + {}^4\sqrt{x}} = \int \frac{4u^3\, du}{1 + u} = 4 \int \frac{u^3}{1 + u} \, du$

$$= \frac{4}{3} u^3 - 2u^2 + 4u - 4\ln/1 + u/ + C$$

$$= \ \underline{4}\ x^{3/4} - 2x^{1/2} + 4x^{1/4} - 4/1 + x^{1/4}/ + C$$

Example: $\int x^3 \sqrt{x^2 + 4}$

Solution: set $u^2 = x^2 + 4$

$x^2 = u^2 - 4$

$2x\,dx = 2u\,du$

therefore $\int x^3 \sqrt{x^2 + 4}\,dx = \int x^2 \sqrt{x^2 + 4}\,.x\,dx$

$= \int (u^2 - 4)\,u.\,u\,dv = \int (u^2 - 4)\,u^2\,du$

$= \int (u^4 - 4\,u^2)\,du$

$= \int \frac{u^5}{5} - \frac{4\,u^3}{3} + C$

$= \frac{(x^2 + 4)^{5/2}}{5} - 4\,\frac{(x^2 + 4)^{3/2}}{3} + C$

$= \underline{1}\;(x^2 + 4)^{3/2}\,(3x^2 - 8) + C$

4.0 CONCLUSION

In this unit you have studied how to obtain reduction formula of certain product functions. Also you have studied how to use appropriate substitution to integrate rational function involving expressions such as sinx, cosx, tanx etc. You have used trigonometric ideitities to integrate product function involving expression like sinbx cosux, sinbx, sincx and cosbx cosax. You have also studied how to integrate rational functions involving fractional powers of the variable x. This unit deals mainly with integration of functions emanating from problems of alternating correct theory, heat transfer, bending of beams, cable stress analysis in suspension bridges, and many other places where trigonometric series is involved.

5.0 SUMMARY:

In this unit you have studied how to:

(1) obtain reduction formula of special integrals such as

(i) $\int \cos^n x\,dx$ (ii) $\int \sin^n x\,dx$

(iii) $\int e^{ax}\cos bx\, dx$ (iv) $\int e^{x} x^{n}\, dx$

(2) integrate rational functions involving expressions such as sinx, cosx, secx tanx and cosecx.

(3) integrate product functions of trigonometric functions such as $\int \cos bx \cos ax\, dx$, $\int \cos bx \sin ax\, dx$ and $\int \sin ax \sin bx\, dx$

(4) evaluate integrals of rational functions involving fractional powers of the variable x.

6.0 TUTOR MARKED ASSIGNMENTS

1. $\int_0^1 x^4 e^x\, dx$
2. $\int \sin x\, dx$
3. $\int \cos^5 x\, dx$
4. $\int e^3 \cos 2x\, dx$
5. $\int e^{x/2} \cos x\, dx$

obtain a reduction formula for

6. $\int \cos^n x\, dx$
7. $\int e^{ax} \sin bx\, dx$
8. $\int \frac{dx}{1 + \cos x}$
9. $\int \frac{\sqrt{x}\, dx}{1 + x^{1/4}}$
10. $\int \frac{\sec x}{2 \tan n + \sec 2-1} dx$
11. $\int \frac{\sqrt{x}\ dx \quad 5}{4(1 + x^{3/4})}$
12. $\int \cos 3x \sin 2x\, du$
13. $\int \cos5 x \cos 3x\, dx.$
14. $\int \cos 4x \cos 7x\, dx$
15. $\int \sin 7n \sin 2x\, d$

UNIT 9

APPLIATION OF DEFINITE INTEGRATION I

TABLE OF CONTENTS

1.0 INTRODUCTION

In unit 1, you studied the connection between sums of thin rectangles of area f(x). Δxr and the integration of f(x). You discovered that when f(x) is continuous on $a \leq x \leq b$ then the limit $\lim \sum f(x)\, \Delta x = F(b)$ as $\Delta x \rightarrow 0$. In unit 1 you applied the above when finding the area under the curve of f(x) within the interval [a,b]. In this unit you shall extend the concept of area under a curve to the following, area between two curves. Distance traveled can be calculated by integrating the velocity $v_1 = f(t)$ of the body. Here $v = f(t) \geq 0$ and continuous function of t within a specified interval of time t. In the next unit calculation of volumes of a solid of revolution will be treated as wellas computing the work done by a body.

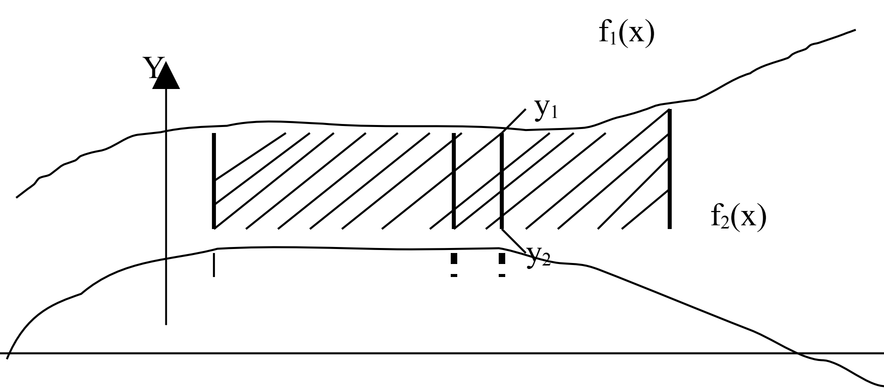
2.0 OBJECTIVES

After studying this unit you should be able to correctly

(i) evaluate area between two curves
(ii) calculate the distance traveled by a body moving with a velocity v = f(t)
(iii) calculate volumes of any solid of revolution

3.0 AREA BETWEEN TWO CURVES

Suppose you consider two continuum functions f(x) and f(x) in a closed interval [a, b]. Suppose that $f_1(x) \geq f_2(x)$ for all $X \in [a, b]$. Then the curve of $f_1(x)$ is always above the curve of $f_2(x)$. (see fig. 9,1).



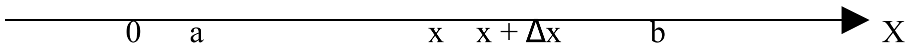


Fig. 9.1

To calculate the area between the two curves you will consider the area under curve f(x) and the vertical lines x = a and x = b then the area above curve $f_2(x)$ and the vertical lines x = a and x = b. To understand this cut out a rectangular STRIP OF WIDTH ΔX. You will not find it difficult to know that the length of the rectangular strip is $y_1 - y_2$. Therefore, the area of rectangular strip is $(y_1 - y_2)\,.\Delta = [f_1(x) - f_2(x)].\Delta x$. Using the concept studied in unit 1, the area under the two curves will be given by the sum of the areas of the rectangular strip i.e.

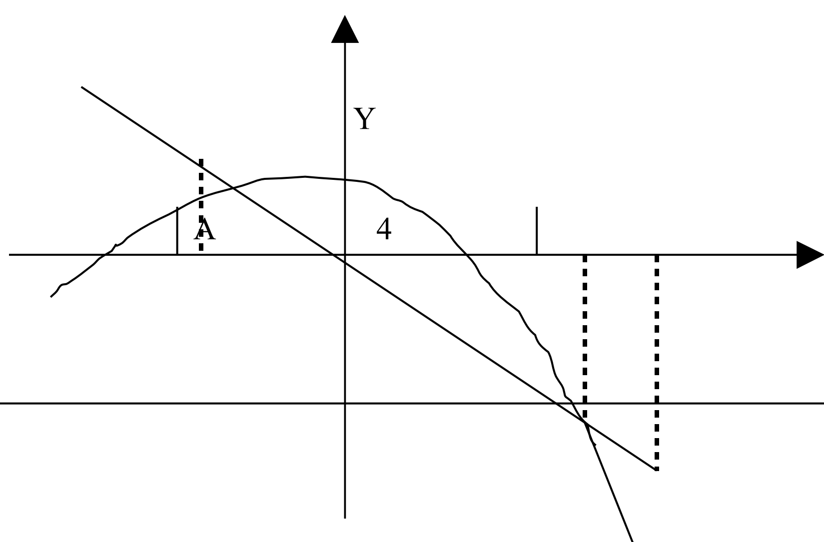
$$A \approx \sum_{a}^{b} [f_1(x) - f_2(x)]\,\Delta x = \int (f_1(x) - f_2(x)\,dx$$

If you allow $\Delta x \rightarrow 0$ then you can obtain the exact area as

$$A = \lim_{\Delta x \rightarrow 0} \sum_{a}^{b} [f_1(x) - f_2(x)]\,\Delta x = \int (f_1(x) - f_2(x)\,)\,dx$$

Example: Find the area bounded by the parabole y = -2x.

Solution: The first step is to know which curve is the upper boundary and which is the lower boundary. This can be achieved by plotting the curves on the same rectangular axes. See fig. 9.2).



X

-10 -1 2 4 10

BC

y = - 3X

$y = 4 - x^2$

Fig 9.2

Second step is to know where the curves intersect. This you can do by finding points that satisfy both equations simultaneously. That is you solve

$$4 - x^2 = -3x$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = 4 \text{ or } -1$$

Thus the curves intersect at A(-1, 3) and B(4, - 12).

For values of $X \in [-1, 4]$ the curve $y = 4 - x^2$ is above the line $y = -3x$ by an amount

$$(4 - x^2) - (-3x) = 4 + 3x - x^2$$

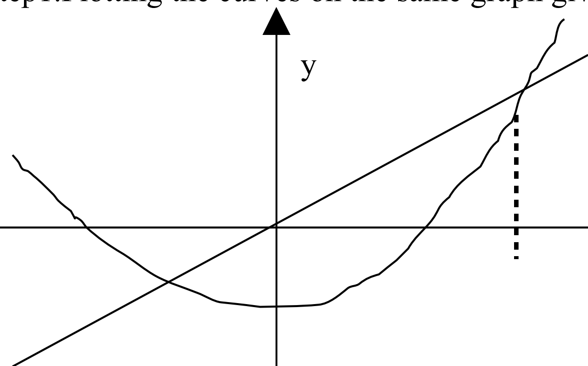
therefore the area between the two curves is given as

$$\int_{-1}^{4} (4 + 3x - x^2)\,dx = 4x + \frac{3x^2}{2} - \frac{x^3}{3} \Big/_{-1}^{4}$$

$$= \frac{125}{6} \text{ sq units}$$

Example: Find the area bounded the parabola $y = x^2 - 2$ and the strainght line
$y = 2x + 1$.

Step1.Plotting the curves on the same graph gives you what you find in Fig. 3



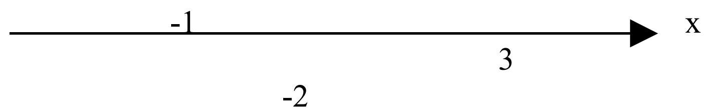


Fig 9.3

Step 2: Point of intersection is given as $x^2 - 2 = 2x + 1 \Rightarrow x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$
$x = 3$ or -1

Step 3: The line $y = 2x + 1$ is above the parabola $y = x^2 - 2$ by an amount
$2x + 1 - (x^2 - 2) = 2x + 3 - x^2$

Therefore the area is given as $\int_{-1}^{3}(2x + 3 - x^2)\, dx = \frac{32}{3}$ sq units

You will now extend the above to finding areas between curves that are crossed. Consider two f(x) and g(x) shown in Fig 9.4

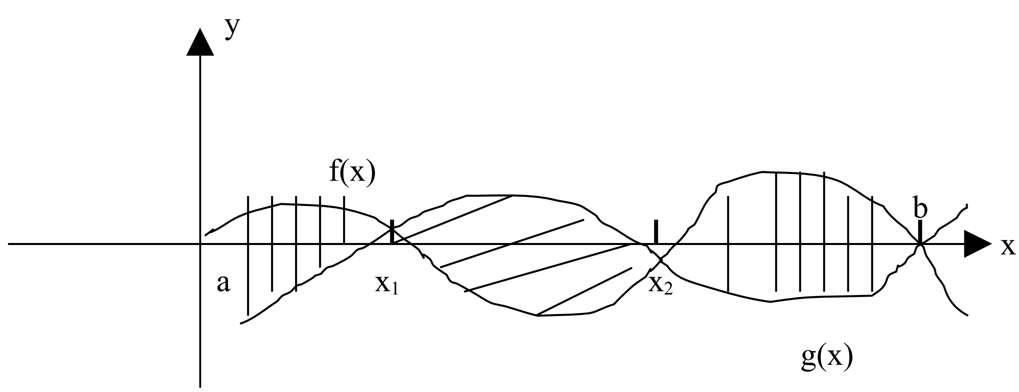


Fig 9.4

In fig 9.4 neither f(x) or g(x) remains positive, i.e. $f(x) > g(x)$ $X \in [a_1\ x_1]$ and $X \in [x_2, b]$ while $g(x) > f(x)$ for $X \in [x_1, x_2]$. Then the area is given as $\int_{a}^{x_1}(f(x) -$

$g(x))\, dx + \int_{x_1}^{x_2}(g(x) - f(x))\, dx + \int_{x_2}^{b}(f(x) - g(x)\, dx = \int_{a}^{x}/f(x) - g(x)/dx + \int_{x_1}^{x_2}/f(x) - g(x)/dx$

Under each integral sign, the upper curve is written first. If you compute just

$$\int^{b} (f(x) - g(x)\, dx$$

the areas will be counted with opposite signs and may cancel out to give you a zero value.

Example: Find the area between $y = \sin x$ and $y = \cos x$ $x \in [\ , 2\pi\ ]$

Solution: The region covered by the area is given in Fig 9.5. The area is given by the integral

$$\int_{0}^{2II} /\sin x - \cos x/\, dx$$

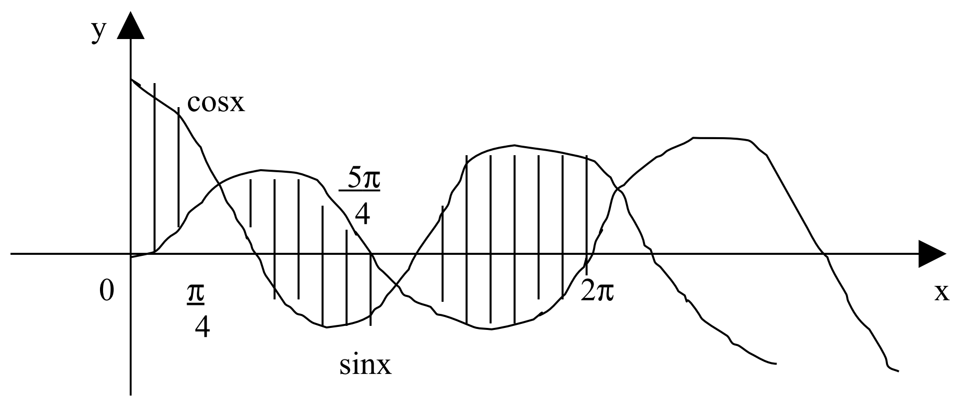


Fig 9.5

We solve for x simultaneously i.e. $\sin x = \cos x \Rightarrow \sin x - \cos x = 0$.

here from the g raph above,

$$\sin x - \cos x \leq 0 \quad x \in [0_{1} \tfrac{\pi}{4}\ ]$$

$$\sin x - \cos x \leq 0 \quad x \in [\tfrac{\pi}{4}, \tfrac{5\pi}{4}\ ]$$

$$\sin x - \cos x \leq 0 \quad x \in [\tfrac{5\pi}{4}, 2\pi]$$

thus

$$\text{area } \int_{0}^{II/4} /\sin x - \cos x\ /dx + \int_{II/4}^{5II/4} /\sin x - \cos x/dx$$

$$+ \int_{\frac{5II}{4}}^{2II} /\sin x - \cos x/dx$$

5II

$$= \int_0^{II/4} (\cos x - \sin x)\, dx + \int_{II/4}^{4} (\sin x - \cos x)\, dx$$

$$+ \int_{5II/4}^{2II} (\cos x - \sin x)\, dx$$

$$= \sin x + \cos x \Big/_0^{II/4} + -\cos x - \sin x \Big/_{II/4}^{5II/4}$$

$$+ \sin x + \cos x \Big/_{5II/4}^{2II}$$

$$= \sqrt{2} - 1 + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2} \text{ sq units}$$

You could also compute areas in which the boundary curves are not functions of x but functions of y (see Fig 9.6). In such cases the area is given by the integral.

$$\int^{b} /f(y) - g(y)/\, dy$$

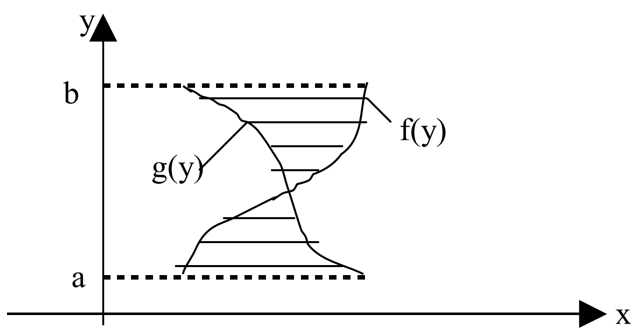
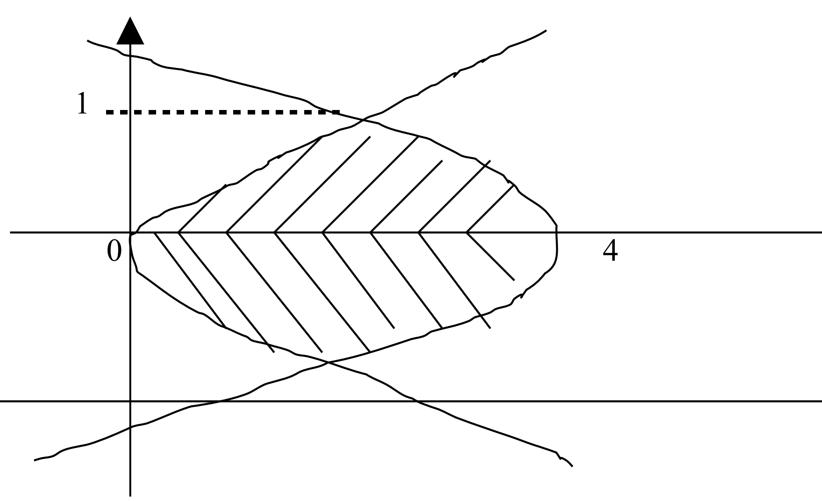


Fig. 9.6

Example: Find the area between the parabola $x = -3y^2 + 4$ and $x = y^2$.

Solution: The region covered by area is displayed in Fig. 9.7. The two parabolas intersect at y = -1 and y = 1 $\Rightarrow -3y^2 + 4 = y^2 \quad -4y^2 = -4$

$y = \pm 1$



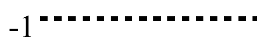


Fig 9.7

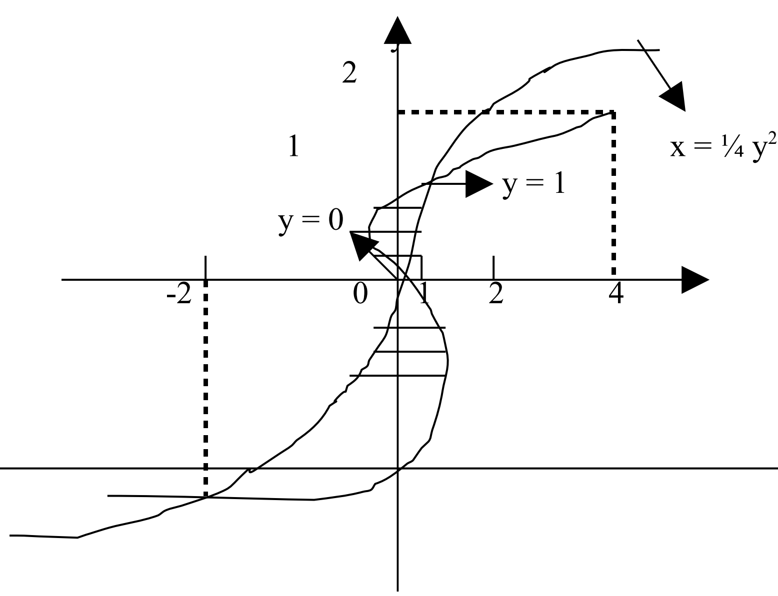
The area is given by the definite integral $A = \int_{-1}^{1} (-3y^2 + 4 - y^2)\, dy$

$$= \int_{-1}^{1} (4 - 4y^2)\, dy = \frac{16}{3}$$

Example: Find the area between

$x = \frac{1}{4} y^3 \quad y \in [-2, 1]$ and
$x = \frac{1}{2} y^3 + \frac{1}{4} y^2 - \frac{1}{2} y \quad y \in [-2, 1]$

Solution: The region bounded by the two curves is disploayed in Fig. 9.8.



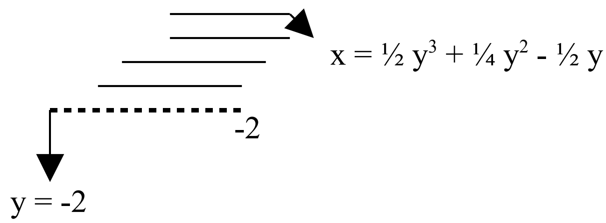


Fig 9.8

The two curves meet at where $\frac{1}{4} y^3 = \frac{1}{2} y^3 + \frac{1}{4} y^2 - \frac{1}{2} y$ x 4

$$y^3 = 2y^3 + y^2 - 2y$$
$$\Rightarrow \quad y^3 + y^2 - 2y = 0$$
$$y (y^2 + y - 2) = 0$$
$$y (y - 1)(y + 2) = 0$$
$$y = 0, y = 1, \ y = -2$$

therefore area is given as

$$\int_{2}^{1} / \tfrac{1}{2} \ y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} y - (1/4 \ y^3)/ \ dx$$

$$= \int_{-2}^{0} / \tfrac{1}{2} y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} y - (1/4 \ y^3)/ \ dx$$

$$+ \int_{0}^{1} / \tfrac{1}{2} y^3 + \tfrac{1}{4} y^2 - \tfrac{1}{2} - (\tfrac{1}{4} y^3)/ \ dx$$

$$= \int_{-2}^{0} (\tfrac{1}{4} y^3 - \tfrac{1}{4} y \ y^2 + \tfrac{1}{2} y) \ dx$$

$$+ \int_{1}^{1} (- \tfrac{1}{4} y^3 - \tfrac{1}{4} y^2 + \tfrac{1}{2} y) \ dx$$

$$= \frac{2}{3} + \frac{5}{48} = \frac{37}{48}$$

Exercises: Sketch the region that is bounded by the following curves and find the area.

1. $y = x^2$, $y = 4x - 3$

2. $\frac{1}{2} y^2 = x$, $x = 4 + y$

3. $x + y^2 - 4 = 0$, $x - 2y = 0$

Ans: (1) $\frac{4}{3}$, (2) 36 (3) $\frac{49}{6}$

3.1 DISTANCE

Let the distance traveled by a body moving with velocity v = f(t) be denoted by the letter S. If f(t) ≥ 0 and continuous in a closed interval t€[a, b]. Then the distance traveled is given as:

$$\int_a^b ds = \int_a^b f(t)\, dt$$

$$S = \int_a^b f(t)\, dt$$

So if you integrate the velocity function you can get the distance traveled by a body.

Example: A boy enters a car at time t = 0. After tsecs, the velocity of the car is $10t^3$m/s. How far does the car move during the first 1 sec?

Solution: Think of the time t = 0 and t = 1 sec. The integral

$$\int_0^1 10t^3\, dt$$

$$= \int_0^1 10t^3\, dt = \frac{10t^{43}}{4}\Big/_0^1 = \frac{5}{2}$$

Example: Find the total distance traveled by a moving body as a function of time. If f(t) = 2t + 1 0 ≤ 2.

Solution: $\int_0^2 f(t)\, dt = \int_0^2 92t + 1)\, dt$

$$= t^2 + t \Big/_0^2$$

$$= 6m$$

Example: A ball is thrown up from the ground at t = 0. At time t its velocity is 30 – 20t m/s. After 3secs, the ball hhits the ground. How far has it traveled?

Solution: In this example, you will encounter a negative velocity which will result to negative distance. This is simply because of the fact that at the maximum height attained the ball falls back towards the ground. The movement backwards is measured as negative velocity and negative distance (see Fig 9.10).

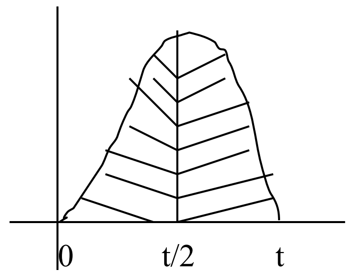


Fig. 10

$v = f(t) \geq 0 \quad t \in [0, t/2]$

$v = f(t) \leq 0 \quad t \in [\, t/2, t]$

Therefore $S = \int_0^t /f(t)/\, dt = \int_0^{t/5} /f(t)/\, dt + \int_{t/2}^t /f(t)/\, dt$

Apply the above to the problem you set $t = 3$, $v = 30 - 20t$

then $\int_0^7 /30 - 20t/\, dt = \int_0^{3/2} (30 - 20)\, dt + \int_{3/2}^3 (20t - 30)\, dt$

$= 30t - 10t^2/_0^{3/2} + 10t^2 - 30t/_{3/2}^3$

$= \frac{45}{2} + \frac{45}{2} = 45\text{m}$

In the above example, $\int_0^3 (30 - 20t)dt = 0$. This is because the ball moves

45/2m with positive velocity (upwards) and 45/2m with negative velocity (downwards). The integral, however, counts these distances with opposite signs so they cancel out.

Exercises: A body has velocity $v = f(t)$. Find the distance covered between $t = a$ and $t = b$.

1. $f(t) = (3t - 1)$m/s $a = 0$, $b = 3$

2. $f(t) = (3t - t^2)$m/s a = 0, b = 2

3. $f(t) = (4t^2 + 3t + 1)$m/s a = 0 b = 3

Ans: (1) $\frac{21}{2m}$ (2) $\frac{9}{2m}$ (3) $\frac{105}{2m}$

4.0 CONCLUSION

You have studied how to use integration to find the areas between two curves. You observed that when the curves crossed one another, the sum of the areas cancel out. As such caution is applied when finding areas bounded by two crossing curves. That is the curve that is above within a given region is ued first in the integral. You have studied how distance traveled by a body with a constant. Velocity can be calculated by integrating the velocity function within a given interval of time. In the next unit, you will apply the same method to finding the work done when an object is moved along a straight line. Also the volumes of a solid of revolution.

5.0 SUMMARY: In this unit you have studied how to:

(i) find the area bounded by two curves
(ii) find the area bounded by two crossing curves
(iii) find the distance traveled by an object with a velocity v = f(t)

6.0 TUTOR MARKED ASSIGNMENT

Sketch the area of the region bounded by the following curves and find the area

1. $y = x^2 - 2$ and $y = 2x + 1$
2. $y = 4 - x^2$ and $y = ½ x + 1$
3. y = cosx and y = sinx XЄ [0, 3_{Π} and t = b
4. $f(t) = 2t - 1$ and a = 0, b = 3
5. $f(t) = t^2 + 1$ and a = 0, b = 2
6. $f(t) = t^2 + \frac{2}{3t + 1}$, a = 0 b = 4
7. $f(t) = 2t - t^2$ a = 0, b = 28 $f(t) = 1 - t^3$ a = 0, b = 1
8. Find the area between

$$x = \frac{1}{8} y^3 + \frac{1}{16} y^2 - \frac{1}{8} y \quad y\in [-1, 1]$$

and $x = \frac{1}{16} y3$ y€ [-2, 1]

9. A particle is put inside an accelerator at time t = 0. After t sec its velocity is $10^5 t^2$m/s. How far does the particle move during the first 10^{-2} sec?

UNIT 10

APPLICATION OF INTEGRATION II

TABLE OF CONTENTS

1.0 INTRODUCTION

In continuation the application of definite integration to specific problems or situation, will in this unit consider further application namely: (1) computing the work done by a force applied along a line. And (ii) volume of a solid. The method that will be adopted in finding the volume of a solid by integration is to slice the solid into numerous thin pieces, each of which is approximately a familiar shape of a known volume. The above could be executed in four steps (1) choose a method of slicing the solid (2) choose a variable x which locates the typical slice and find the range of values of x that applies to the problem. (3) compute the volume f(x) dx of a typical slice and finally (4) find anti-derivative (integration) of f(x) and compute $\int_a^b f(x)dx$ where x€[a, b].

The four steps enumerated above will be useful in finding volumes of any solid by integration.

2.0 OBJECTIVES: After studying this unit, you should be able to:

(i) compute the work done by applying a force on an object alng a line.
(ii) Compute the volume of a solid.

3.0 WORK

When a constant force F(N) is applied along a distances in the work done, (Nm) is the product of force and distance. i.e. Work = W = F.s suppose an object is moved along a straight line fro x = a to x = b by a force of magnitude f(x). Dividing the interval [a, b] into sub-intervals of length Δx then the work done moving the object from $xe i_{-1}$ to xi is approximately f(xl).Δxi since Δx0 this force is constant. The total work done will then be the sum of work done in each subinterval. This is given as:

$$\text{Total Work} = \int_a^b f(x)\,dx$$

Example: Suppose at each point of the x – axis there is a force of $(3x^2 - x + 1)$N pulling an object. Find the work done in moving it from x = 1 to x = 3.

Solution: Work = $\int_1^3 (3x^2 - x + 1)\,dx = x^3 - U\,x\,U + x/_1^3$

$= 24$

An interesting example of this can be derived from Newton's Law of Gravitation.

Example: Given that two bodies pull at each other with a force $F = k\,\frac{Mm}{x^2}$ where M and m afd their masses respectively and x is distance between Them. If one of the bodies is fixed at origin, how much work is done in moving the other body from x = 1 to x = 3? (Assume k, M, and m are known)?

Solution:

$$W = \int f(x)\,dx = \int_1^3 \frac{kMn}{x^2}\,dx$$

$$= kMm\int_1^3 \frac{1}{x^2}\,dx = -kMm\,\frac{1}{x}\Big/_1^3 = \frac{2}{3}\,kMn.$$

For most springs, there is a law governing their functions. According to the law known as Hooke's law when a spring is stretched a short distance there is a compressing or restoring force proportional to the amount of stretching. This force is given as F = cx where x is the amont the spring has been displaced from its natural or unstressed length and c is a spring constant. Beyond this range, the spring will become unreliable and unpredictable.

Example: A spring has a natural length of L = 0.20m. A force of 1N streches the spring to a length of 0.21m. Find the spring constant. Find the amount of work required to stretch the spring from its natural length to a length of 0.22m. Calculate the amount of work done in stretching the spring from 0.21m to 0.22m.

Solution: F = 1, extra length = .20 – 21 = 0.1m
then F = cx ⇨ 1 = c(0.1)
c = 1/0.1 = 10.

To calculate the work done in stretching the spring 0.02m beyond it natural length, you have

$$W = \int_0^{0.02} 10x\,dx = x^2/_0^{.02} = 2 + 10^{-3}N$$

To find the work done in stretching the spring from a length of 0.21m to a length of 0.22m you compute

$$W = \int_{0.01}^{0.02} 10x\, dx = 1.5 \times 10^{-3} N.$$

Example: Find the work done by a force f(x) = 2x + 3 N in moving an object from x = 1m to x = 5m.

Solution: $W = \int_1^5 f(x)\, dx = \int_1^5 (2x + 3)dx = 36$

Exercises:

1. The force in Newton's required to stretch a certain spring xm is given as F = 5x. How much work is required to stretch the spring (1) 0.1m (ii) 0.15m and (iii) 0.025m?

2. Find the work done by a force f9x) = 7x + 5 Newton in pushing an object from x = 1m to x = 2m.

3.1 VOLUMES

In this section you will study how to find the volumes of solids by integration. Before doing this, you will compute the area of circle and then extend the same methd to that of computing the volumes of solids.

Examples: Find the area of a circle of radius r. Where the circumference is given as C = 2Лr.

Solution: Step 1. Cut the circle into this concentric rings (Fig 10.0)

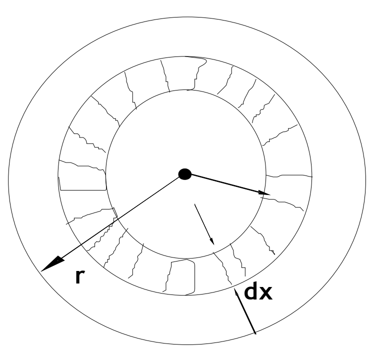


Fig. 10.0

Step 2: Let x denote the distance of a ring from the centre as shown in Fig 10.1 here $0 \leq x \leq r$.

Step 3: The length of any ring is given as 2Лx and the width is dx. Thus the area of a typical ring is given as 2лx dx. The area of the circle is therefore given as the sum of areas of the concentric rings.

i.e. $A = \int_0^r 2Лx\, dx.$

$= Лx^2/^r = Лr^2$ which is the area of a circle with radius r.

Volume of Revolution: The volumes of many solids can be found by the method of slicing describd above. Before you continue it is necessary to define what is meant by solid of revolution A solid which has a central axis of symmetric is called a solid of revolution. Most solid that will be consider in this unit are solid of revolution. For example, a cone, a cylinder, a bucket etc. To find the volume of such solid displayed in fig. 10.1 you irst consider the area under the region AB of the curve y = f(x) revolved about the x - axis

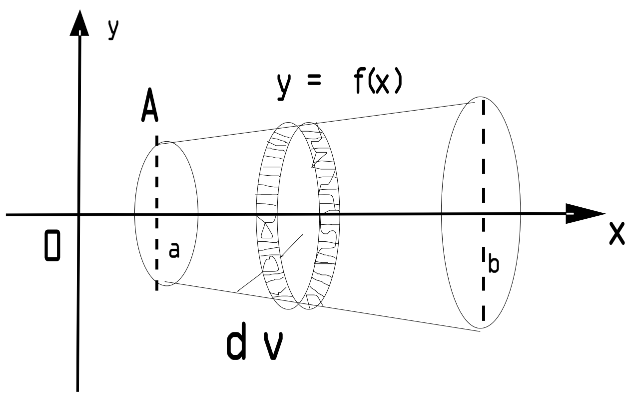
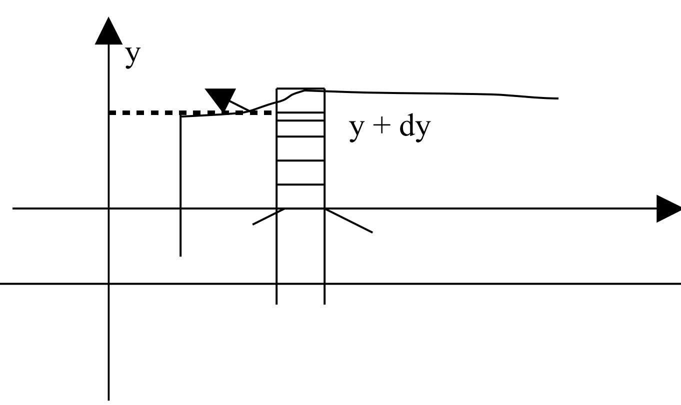


Fig 10.1

through 360^0. Each point on the curve represents a circle with centre on the x-axis. A solid of revolution as displayed in Fig 10.1 hs two circular plane ends cutting the x-axis at x = a at x = b. As ws done in the example of the area of a circle. Let v be the volume of the solid of revolution from x = a up to any value XЄ (a, b) see Fig 10.2



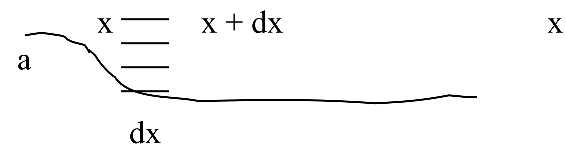


Fig 10.2 Showing a cross section of solid of revolution

Suppose thee is an increment in x i.e. dx with a corresponding increment in y, dy and increment in V, dv. In fig 10.2 he volume of the slice with thickness dx is given by dv and is enclosed between two cylinders of outer radius y = dy and inner radius y.

Thus $\text{Л}y^2\, sx = sv \leq \text{Л}\,(y + sy)^2\, dx$

$\Rightarrow \text{Л}y^2 \leq \frac{sv}{sx} \leq \text{Л}(y + sy)^2$ ___ (A)

hence as $sx \rightarrow 0$, $sx \rightarrow 0$ and $\frac{sv}{sx} \rightarrow \frac{dv}{dx}$

thus (A) becomes

$$y^2 \leq \frac{dv}{dx} \leq \text{Л}y^2$$

$\Rightarrow$ $\frac{dv}{}$ = $\text{Л}y^2$ integrating both sides you have

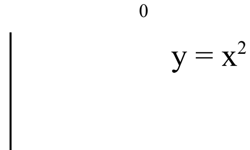
$$V = \int \text{Л}y^2\, dx$$

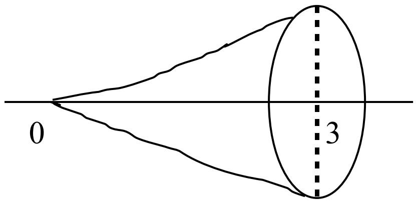
where y = f(x) a continuous function and V is the volume of solid generated when the curve y = f(x) is rotated through 360^0 around the x-axix between the limits x = a and x = b.

Example: The region in the x-y plane bounded by the curve $y = x^2$, the line x = 0 and x = 3 and the x = axis is revolved about x – axis. What is the resulting volume?

Solution: See Fig (10.3)

$$V = \int_0^3 \text{Л}y^2\, dx$$





$$V = \int_0^3 Л\, x^4 dx$$

$$= \frac{Лx^5}{5}\Big/_0^3 = \frac{243Л}{5}$$

Fig 10.3

Example: Let the region be revolved around the y –axis from x = 0 to x = 2. What is its volume?

Solution: See Fig 10.4

$$V = \int^2 Лx^2\, dy$$

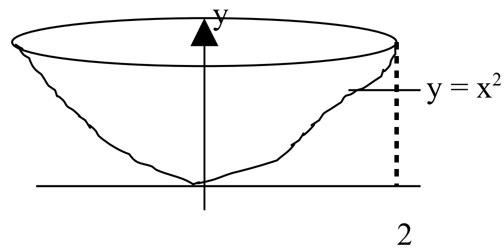


Fig. 10.4

Since $y = x^2$ thus implies that for x = 0, y = 0 and x = 2, y = 4 then

$$V = \int_0^4 Лy\, dy = Л\, \frac{y^2}{2}\Big/_0^4 = 8\, Л$$

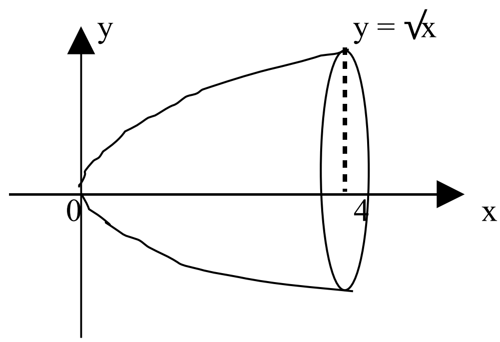
Example: The portion of the curve $y = \sqrt{x}$ between x = 0 and x = 4 is rotated completely round the x-axis. Find the volume of the solid generated.

Solution: See Fig. 10.5

$$V = \int_0^4 Лy^2\, dx$$

$$= \int_0^4 Лx\, dx$$

$$= Л\, \frac{x^2}{2}\Big/_0^4 = 8л$$



Example: Find the volume of a cone with base radius r and height h.

Solution: The axis of the cone is a line that passes through the vertex of the cone and the centre of the base.

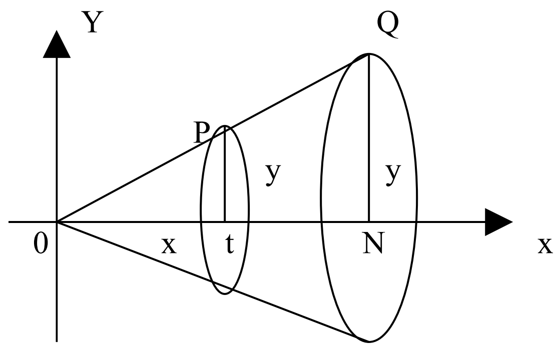


Fig. 10.5

$$V = \int_0^h Л\, y^2\, dx$$

To get y^2 you consider Fig. 10.5 where a cross section gives you a picture of two similar triangles Δ0QN and Δ OPT. Thus

$$\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{rx}{h} \Rightarrow y^2 = \frac{r^2 x^2}{h^2}$$

therefore $V = Л \int_0^h \frac{r^2 x^2}{h^2} dx = \frac{Лr^2}{h^2} [x^3]_0^h = \frac{1}{3} Лr^2h$

which is te volume a circular cone.

Example: Find the volume of a sphere by rotating the circle x^2 $ty^2 = r^2$ about the x-axis.

Solution: Using the formula

$$V = \int_0^b лy^2\, dx \,.\, Y$$

since the centre of the circle is at origin $y = \sqrt{r^2 - x^2}$ $-r \leq x\ r$

then a = r , b = r

$$V = \int_{-r}^{r} Л(r^2 - x^2)\, dx = Л \left[r^2x^2 - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3} Лr^3$$

Exercises: Sketch the graph and find the volume generated by revolving the region below it about he x-axis.

1. $y = 3x^2$ XЄ (0, 1)
2. $y = e^{-x}$ XЄ (0, 2)
3. $y = 4x^3$ XЄ (0, 1)
4. $y = 1/x$ XЄ (1, 2)
5. Sketch the graphs and find the volume generated by revolving the region between them about the x-axis $y = 4-x^2$ $y = ½ x + 1$

Ans:

1. $\frac{9}{5Л}$ (2) $\frac{Л(1 – e^{-4})}{2}$ (3) $\frac{16Л}{7}$

4. $\frac{Л}{2}$ (5) $\frac{2401}{80}$

3.2 AVERAGE VALUE OF A FUNCTION

You are quite familiar with how to find average value of a finite number of data. For example, if $y_1, y_2,\ldots, y_n$ are scores obtained in a class test by n number of students, then the class average score will be given as

$y_{av} = \frac{y_1, + y_2 + y_n}{n}$ – 1

When the number is infinite then it will not be possible to use the above formula (1). The above is also possible for a discrete case. If data is continuously used in a given interval, formula 1 will be difficult or meaningless. In such situation another method is needed to be able to calculate the average value of the data y.

Let $y = f(x)$ $a \leq XЄ\ b$ then $Yav = \frac{1}{b-a}\int_a^b f(x)\,dx$

Example: Find the average value of $[\ 0_1\ a]$ if $f(x) = x^3$

Solution:

$$Yav = \frac{1}{b-a}\int_a^b f(x)\,dx,\ a = 0,\ b = a$$

$$= \frac{1}{a}\int_0^a x^3\,dx = \frac{1}{a}\left[\frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{a}\ \frac{a^4}{4} = \frac{a^3}{4}$$

Example: Find the average value of $f(x) = \sqrt{r^2 - x^2}$ $X \in [-r, r]$

Solution: $a = r$, $b = r$, $f(x) = \sqrt{r^2 - x^2}$

Therefore: Average = $\frac{1}{} \int^{r} (r^2 - x^2)^{1/2} dx$

$$= \tfrac{1}{2} r \left[\tfrac{1}{2} x \sqrt{r^2 - x^2} + \tfrac{1}{2} r^2 \arcsin \frac{x}{r} \right]^{r}$$

$$= \tfrac{1}{2} r \left[\tfrac{1}{2} r^2 \pi \right] = \frac{r\pi}{4}$$

4.0 CONCLUSION

In this unit you have studied how to find the work done when a force is applied on an object along a straight line. You have studied how to compute the work done when a spring is compressed or stretched. You have studied how to compute volumes of a solid generated by revolving a region along the axis of symmetry of the solid. You have also studied how to find the average value of a set of continuous data in a gien interval.

5.0 SUMMARY: In this unit you have studied how to:

(i) compute the work done when a spring is compressed or stretched

i.e. $W = \int_a^b f(x)\,dx$

(ii) compute the volumes of solid of revolution

i.e. $V = \int_{x=a}^{x=b} \pi y^2\,dx$ or $V = \int_{y=a}^{y=b} \pi x^2\,dy$

(iii) compute the average value of a function f(2) i.e.

Average $f(2) = \frac{1}{b-a} \int_a^b f(x)\,dx \quad a \le x \le b$

6.0 TUTOR MARKED ASSIGNMENT

1. A certain spring exerts a force of 0.5N when stretched .33m beyond its natural length. What is the work done in stretching the spring 0.1m beyond its natural length? What is the work done in stretching it an additional 0.1m?

2. A hemispherical oil tank of radius 10m is being pumped out. Find the work done in lowering the oil level from 2m below the top of the tank to 4m below the top of the tank. Given that the pump is placed right on top of the tank. Take the weight of water wkg.

3. The base of a solid is the region between the curves $\sqrt{x} + \sqrt{y} = 1$ and $y = 1 - x$. Sketch the graphs and find the volume of the solid generated by revolving the region about the x-axis.

4. Find the volume generated when the plane figure bounded by $y = 5 \sin 2x$, the x-axis and the ordinates $x = 0$ and $x = \pi/4$ rotates about the x-axis through a complete revolution.

5. Suppose a supermarket receives a consignment of 1400 satchets of pure water every 30 days. The pure water is sold to retailers at a steady rate; and x days after the consignment arrives, the inventory I(x) of satchets still on hand is $I(x) = 1400 - 14x$. Find the average daily inventory.