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SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 417

COURSE TITLE: Electromagnetic Theory

ELECTROMAGNETIC THEORY

James Clerk Maxwell (1831-1879)

NATIONAL OPEN UNIVERSITY OF NIGERIA

MTH 417 – Electromagnetic Theory

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Topics:

Maxwell's field equations.
Electromagnetic waves and theory of lights.
Plane electromagnetic waves in non-conducting media,
reflection and refraction of plane boundary.
Wave guide and resonant cavities.
Simple radiating systems.
The Lorentz-Einstein transformation.
Energy and momentum.
Electromagnetic 4-vectors.
Transformation of (E.H.) fields.
The Lorentz force.

Preface

These lecture notes treat the mathematical theory of electromagnetism. They are written at a level appropriate for undergraduate and graduate students in mathematics. Very little is assumed about prior exposure to the physical theory, but some mathematical sophistication is assumed at various points in the exposition. The subject is treated as a continuum theory with only brief mention of underlying molecular origins of phenomena. Also, thermodynamical considerations are not emphasized.

The purpose of this piece is to give a self contained treatment of electromagnetism for students of mathematics in order to lay the foundation for the many applications of these ideas in applied mathematics.

Maxwell's Equations

I. Introduction:

The basic equations of electromagnetism are the four Maxwell Equations and the Lorentz force law. In principle these, together with Newton's second law of motion are enough to completely determine the motion of an assembly of charges given the initial positions and velocities of all the charges. It is well known that light is a form of electromagnetic radiation, so it is instructive to review some of the properties of electricity and magnetism leading to the derivations of the Maxwell's equations.

The original studies of electricity and magnetism date back to at least the early Greek times. By the start of the nineteenth century, it was known that some objects could possess an **electrical charge**, and that these charges could exert a force on each other even through a vacuum. This force could be described mathematically as

$$\vec{F}_E = q\vec{E}, \tag{1.1}$$

where q is the electrical charge on the object in question and $\vec{E}$ is the **electric field** produced by all the other charges in the universe. The charge was discovered to take on a discrete set of values, one of the first examples of quantization. In its turn, the electric field can be described by a scalar potential field V, which is related to the electric field by

$$\vec{E} = -\nabla V. \tag{1.2}$$

The vector, differential-operator ∇ in these equations is defined as

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

In addition, it was also noted that a moving charge may experience another force which is proportional to its velocity $\vec{v}$. This led to the definition of another field; namely the magnetic field $\vec{B}$, such that

$$\vec{F}_B = q\vec{v}\times\vec{B}. \tag{1.3}$$

As with the electric field, the magnetic field is generated by all the other currents in the universe. The magnetic field can be described in terms of a vector potential field $\vec{A}$, which is related to the magnetic field by

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{1.4}$$

The Lorentz force law

We now begin to consider how things change when charges are in motion[1]. A simple apparatus demonstrates that something wierd happens when charges are in motion: If we run currents next to one another in parallel, we find that they are attracted when the currents run in the same direction; they are repulsed when the currents run in opposite directions. This is despite the fact the wires are completely neutral: if we put a stationary test charge near the wires, it feels no force.

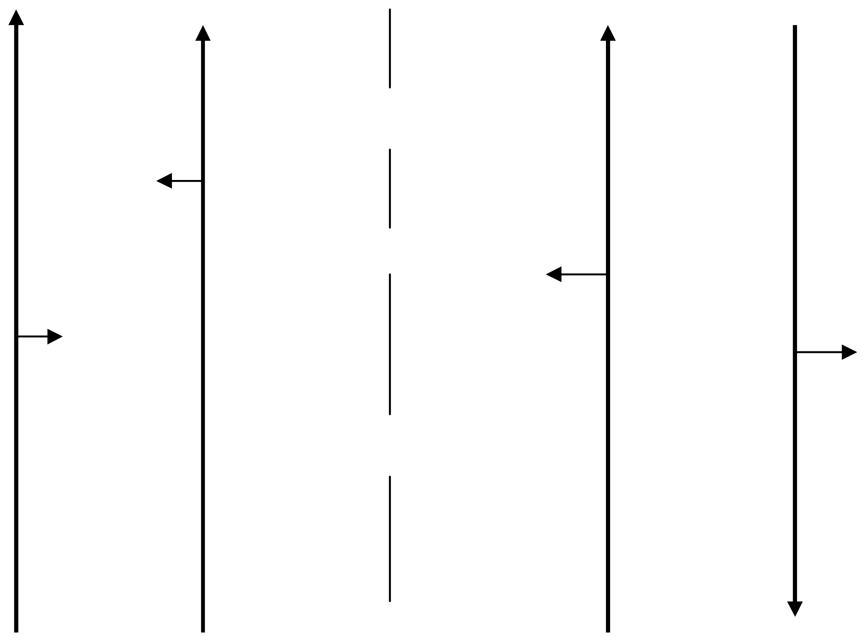

Figure 1: Left: parallel currents attract. Right: Anti-parallel currents repel.

Furthermore, experiments show that the force is proportional to the currents - double the current in one of the wires, and you double the force. Double the current in both wires, and you quadruple the force. This all indicates a force that is proportional to the velocity of a moving charge; and, that points in a direction perpendicular to the velocity. These conditions are screaming for a force that depends on a cross product. What we say is that some kind of field $\vec{B}$ the “magnetic field" - arises from the current. The direction of this field is kind of odd: it wraps around the current in a circular fashion, with a direction that is defined by the right-hand rule: We point our right thumb in the direction of the current, and our fingers curl in the same sense as the magnetic field(Figure 2).

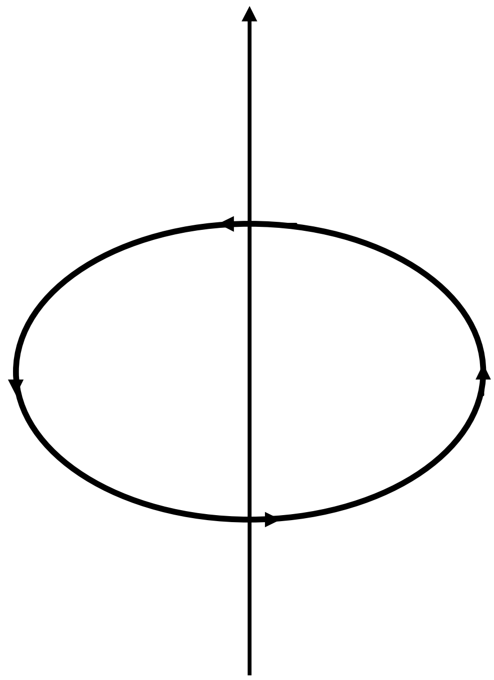

Figure 2:

With this sense of the magnetic field defined, the force that arises when a charge moves through this field is given by

$$\vec{F} = q\frac{\vec{v}}{c}\times\vec{B},$$

where c is the speed of light. The appearance of c in this force law is a hint that special relativity plays an important role in these discussions.

If we have both electric and magnetic fields, the total force that acts on a charge is of course given by:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right). \qquad (1.5)$$

This combined force law is known as the Lorentz force.

The Constitutive relations

Similar to the constitutive relations in continuous medium mechanics, there are also constitutive relationships in electromagnetics. Constitutive relations describe the medium's properties and effects when two physical quantities are related. It cane be viewed as the description of response of the medium as a system to certain input. For example, in continuous medium mechanic, the response of a linear-elastic medium to strain can be described by the Hooke's law, and the resultant is the stress. The relation between stress and strain is the Hooke's law. In another word, Hooke's law is the constitutive relations for linear elasticity. In electromagnetics, there are four fundamental constitutive relationships to describe the response of a medium to a variety of electromagnetic input. Two of them describe the relationship between the electric field $\vec{E}$ and the conductive current $\vec{J}$, and the electric displacement $\vec{D}$, and the other two describe the relationship between the magnetic field $\vec{H}$ and the magnetic induction $\vec{B}$, and the magnetic polarization $\vec{M}$. Quantitatively, these four constitutive relationships are

$$\vec{J} = \sigma \vec{E} \quad \text{(Ohm's law)} \qquad \text{(i)}$$

$$\vec{D} = \varepsilon \vec{E} \qquad \text{(ii)} \qquad (1.50)$$

$$\vec{B} = \mu \vec{H} \qquad \text{(iii)}$$

$$\vec{M} = \chi \vec{H} \qquad \text{(iv)}$$

where σ is the electric conductivity, ε the dielectric permittivity, μ the magnetic permeability, and χ the magnetic susceptibility. It is possible to discuss the electromagnetic properties of earth material in terms of these four parameters. It is noteworthy that the first relation is the well-known Ohm's law in a microscopic form. These four parameters exclusively describe the electromagnetic properties of a material. It is necessary to point out that some of them are inter-related(to be seen later). To understand the behavior of these electromagnetic parameters are the central piece to understand the geophysical response when geophysical surveys are employed to solve any engineering, exploration, and environmental problems.

Maxwell's Equations

The fact that the electric field was described in terms of stationary charges, while the magnetic field was described in terms of moving charges led people to suspect that some relationship existed between the two fields. This was confirmed when it was found that an electric current could be generated by changing the magnetic field. In the mid-1800's, the theories of electricity and magnetism were finally united by James Clerk Maxwell in four equations now known as **Maxwell's equations**.

$$\oint\!\!\oint \vec{E} \cdot d\vec{S} = \iiint \frac{\rho_f}{\varepsilon} dV \,, \qquad (1.6)$$

$$\oint\!\!\oint \vec{B} \cdot d\vec{S} = 0 \,, \qquad (1.7)$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} \,, \qquad (1.8)$$

$$\oint \vec{B} \cdot d\vec{l} = \mu \iint \vec{j}_f \cdot d\vec{S} + \mu\varepsilon \frac{d}{dt} \iint \vec{E} \cdot d\vec{S} \,. \qquad (1.9)$$

Each one of these can be understood separately.

The first of Maxwell's equations, equation (1.6), is known as **Gauss's Law**. It relates the flux of electric field intensity to the total charge enclosed by the surface. The flux is defined as

$$\Phi_E = \oint\!\!\oint \vec{E} \cdot d\vec{S} \,, \qquad (1.10)$$

where ***dS*** is the vector outwardly normal to the surface and the integral is over the entire surface enclosing the region in question. In words, Gauss's law tells us that the total flux through a closed surface, i.e. the change in the number of field lines passing through a closed surface, is proportional to the total charge contained within the volume defined by the surface. Thus if there is no charge inside the surface, the net flux is zero. If there is a positive net charge, the enclosed region acts as a **source**; if the net charge is negative, the enclosed region acts as a **sink**.
The constant ε is called the **electric permittivity** of the medium. If the medium is a vacuum, then $\varepsilon = \varepsilon_0$, where ε_0 is known as the **permittivity of free space** and has a value of $\varepsilon_0 = 8.8542 \times 10^{-12} \ \frac{C^2}{N \cdot m^2}$. The electric permittivity was originally used to act as a medium dependent proportionality constant that connects a parallel plate capacitor's capacitance with its geometric characteristics. Conceptually, we can view the permittivity as encompassing the electrical behavior of the medium: in a sense, it is a measure of the degree to which the material is permeated by the electric field in which it is immersed. We can relate the electric permittivity to the dielectric constant by the following formula

$$\varepsilon = K_e \varepsilon_0. \qquad (1.11)$$

The second equation is also a form of Gauss's law, this time applied to the magnetic field. The fact that the enclosed charge is zero tells us that, at least according to classical electromagnetic theory, there is no such thing as a magnetic monopole. In other words, whereas the electrical charge could be viewed as either a positive or negative charge individually, we can never find magnetic charges which do not include both a positive and negative pole. Since the total enclosed charge is the algebraic sum of the charges, this lack of magnetic monopoles automatically insures that the sum is zero.

The third equation is known as **Faraday's law**. In a manner similar to the electric flux, the magnetic flux is defined as

$$\Phi_B = \iint \vec{B} \cdot d\vec{S}, \qquad (1.12)$$

where the surface is now an open surface bounded by a conducting loop. Faraday found that if the induced emf(electromotive force) that was developed in the loop depended on the rate at which the magnetic flux changed,

$$\text{emf} = -\frac{d\Phi_B}{dt}. \qquad (1.13)$$

However, the emf exists only as a result of the presence of an electric field, which is related to the emf by

$$\text{emf} = \oint \vec{E} \cdot d\vec{l}\ . \qquad (1.14)$$

Combining (1.13) and (1.14), any direct reference to the induced emf is removed and we get Faraday's law. Physically, this shows us that if the magnetic flux changes, in other words if either the surface area or the magnetic field changes with time, then an electrical field is produced as result. This electrical field creates an emf which acts in such a way as to resist the changes in the magnetic flux. Thus, a **time varying magnetic field creates an electric field**. Since there are no charges which act as a source or a sink, the field lines close on themselves, forming loops.

The last of Maxwell's equations is known as **Ampere's Law**. In its original form as expressed by Ampere, it related the number of magnetic field lines which passed through a surface formed by a closed loop to the total amount of current which was enclosed by the loop

$$\oint \vec{B} \cdot d\vec{l} = \mu \iint \vec{j} \cdot d\vec{S}\,, \tag{1.15}$$

where $\vec{j}$ is known as the current density. The open surface is bounded by the loop, and the quantity μ is called the **permeability** of the medium. In a vacuum, $\mu = \mu_0$, where μ_0 is called the **permeability of free space** and has a value of $\mu_0 = 4\pi \times 10^{-7}\ \frac{\mathrm{N \cdot s^2}}{\mathrm{C^2}}$. We can relate the permeability of free space with the permeability via the equation

$$\mu = K_B \mu_0\,, \tag{1.16}$$

where K_B is called the relative permeability. In a manner similar to the dielectric constant, the relative permeability can be viewed as a measurement of how well the magnetic field permeates a material.

While Ampere's law in its original formulation explained many important effects, such as the operation of a solenoid, it was found to also create larger problems. In particular, use of Ampere's law in the form of equation (1.15) led to violation of conservation of energy for the electric and magnetic fields. In order to correct this, Maxwell hypothesized the existence of an additional current, the **displacement current**, which is defined as

$$i_d = \varepsilon \iint \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}\,. \tag{1.17}$$

When this is combined with Ampere's law in a region with no physical currents, we get

$$\oint \vec{B} \cdot d\vec{l} = \mu\varepsilon \frac{d\Phi_E}{dt}\,.$$

In other words, just as a time varying magnetic flux lead to the creation of a circular electric field, so to does a time varying electric flux lead to the creation of a linear magnetic field. If a physical current also exists, we again regain the last of Maxwell's equations.

Differential Form of Maxwell's Equation

In this section we derive the Maxwell equations based of the differentiation form of a number of physical principles. Thus we recast Maxwell's equations into a differential form. This form will be necessary later when we begin discussing the wave nature of light. In order to do this conversion, we first need two important results from vector calculus, **Gauss's divergence theorem** and **Stokes theorem**. Gauss's divergence theorem tells us that the net flux of a vector field through a closed surface is equal to the integral of the divergence of that field over the volume contained in the surface(i.e conversion of integration over s closed Surface to Volume Integral)

$$\oiint \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{F} dV\,. \tag{1.18}$$

Similarly, Stokes theorem states that the flux through a closed loop is equal the integral of the curl of the field over the area enclosed by the loop(integral overs a Closed curve to Surface integral)

$$\oint \vec{F} \cdot d\vec{l} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} . \tag{1.19}$$

Let's start with Gauss's divergence theorem and apply it to the first two of Maxwell's equations. Then we get

$$\begin{aligned} \iiint \frac{\rho}{\varepsilon} dV &= \oiint \vec{E} \cdot d\vec{S} \\ &= \iiint \vec{\nabla} \cdot \vec{E} dV \end{aligned}$$

and

$$\begin{aligned} 0 &= \oiint \vec{B} \cdot d\vec{S} \\ &= \iiint \vec{\nabla} \cdot \vec{B} dV \end{aligned} .$$

These relations must be equal for any volume, so the first two Maxwell's equations in MKSA system become

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{1.20}$$

and

$$\vec{\nabla} \cdot \vec{B} = 0 . \tag{1.21}$$

From the above equations, (1.20) implies that electric charges whose density ρ is are the sources of the electric field $\vec{E}$, while (1.21) implies that Field lines of $\vec{B}$ are closed, which is equivalent to the statement that there are no magnetic monopoles(Figure 3a).

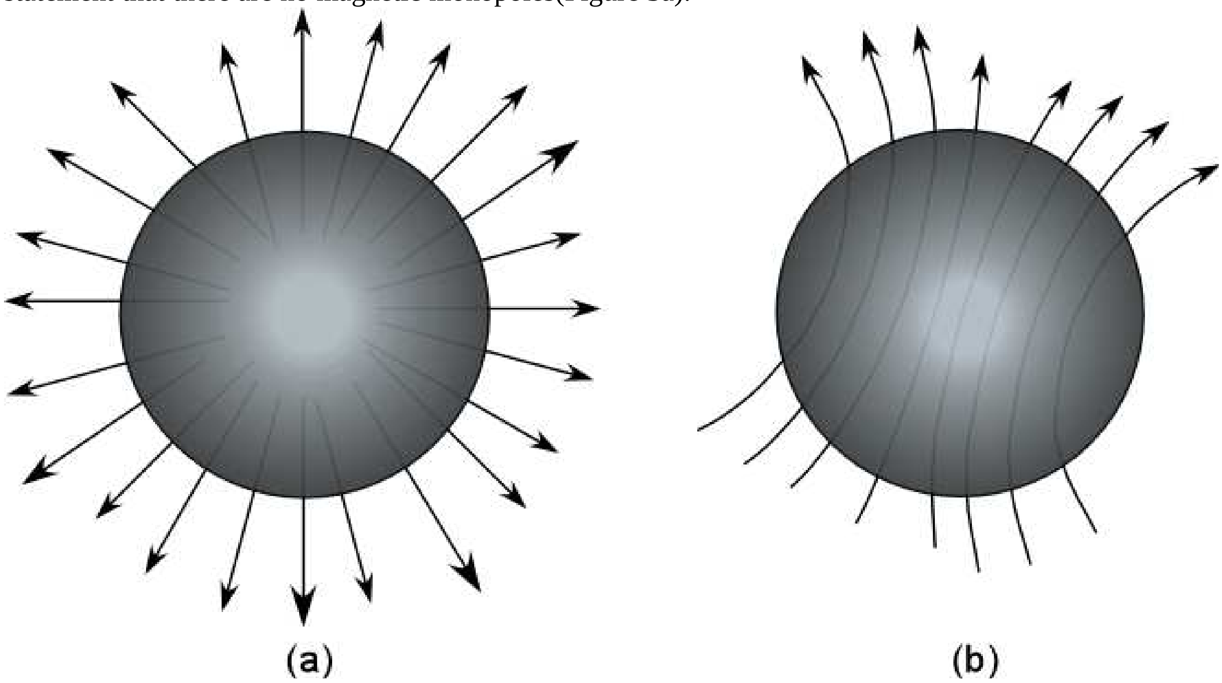

Figure 3. The case of a rotation-free vector field (a) and a source-free vector field (b).

We shall now obtain the last two Maxwell's equations using the Stokes theorem.First, we discuss the Ampere's law. Ampere's law describes the fact of that an electric current can generate an induced magnetic field. It states that in a stable magnetic field the integration along a magnetic loop is equal to the electric current the loop enclosed. Mathematically, Ampere's law can be expressed as:

$$\oint \vec{H} \cdot d\vec{l} = \vec{j}\ . \qquad (1.22)$$

Let us take a simple case to illustrate the Ampere's law, as shown in Figure 2. Recall that the curl of a vector field is defined as

$$Cur\vec{H} = \nabla \times \vec{H} = \lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot d\vec{l}}{\Delta S} \vec{n}. \qquad (1.23)$$

Consider the case of that the magnetic field is on the plane of the paper and the electric current is flowing out from the paper with the current normal to the paper we can have

$$Cur\vec{H} = \nabla \times \vec{H} = \lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot d\vec{l}}{\Delta S} \vec{n}. = \frac{\vec{j}}{\Delta S} = \vec{J}\ , \qquad (1.24)$$

where $\vec{J}$ is the current density in an infinitesimal area. Meanwhile, if the electric field $\vec{E}$ is not stable, i.e., varying with respect to time, and the variation frequency is high enough and extends into the radar frequency, there will be another current in the medium known as the displacement current and is proportional to the variation of the electric fined $\vec{E}$, and the proportional factor is the dielectric permittivity ε. Thus, there will be another contributor, $d\vec{D}$/dt, to induce the magnetic field $\vec{H}$. The displacement current works exactly the same way as the conductive current $\vec{J}$, so that the total current should be $\vec{J}$ + $d\vec{D}$/dt; put both contributors into the above equation ends up with the first equation of the Maxwell's equations:

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad (1.25)$$

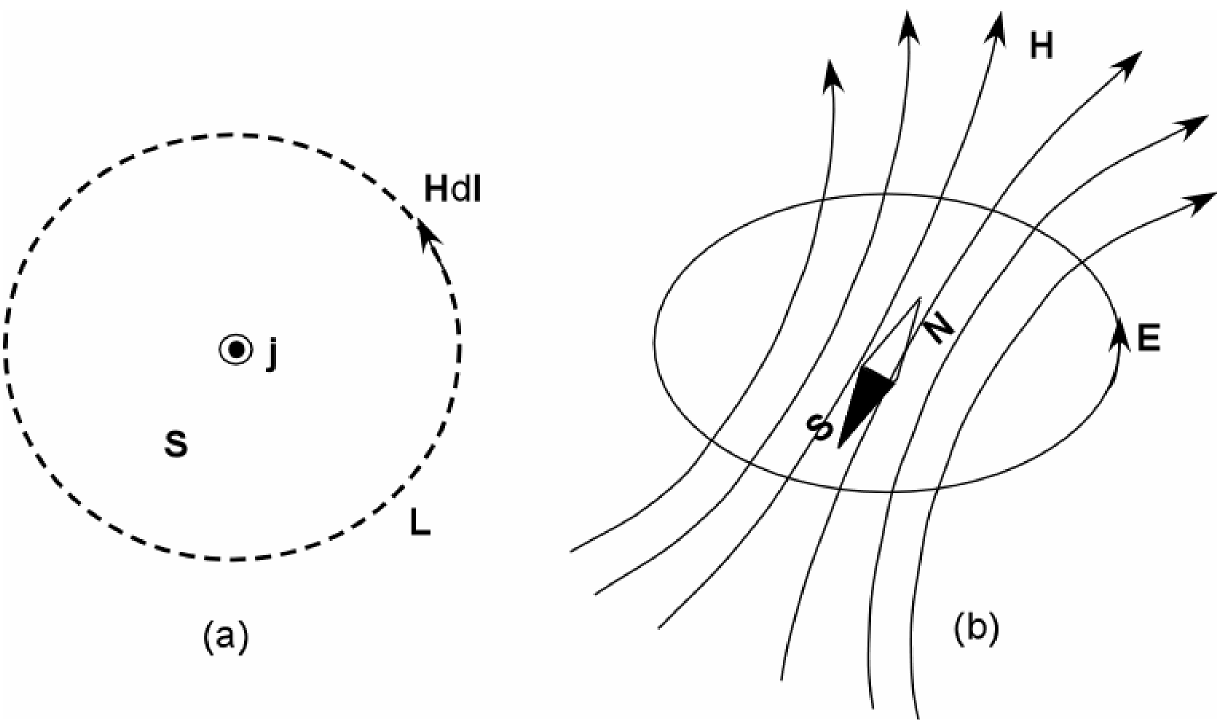

Figure 4. Illustration of the Ampere's law (a) and the Faraday's law (b).

Second, we take a look of the Faraday's law. Faraday's law states that a moving magnet can generate an alternating electric field. Mathematically, the moving magnet can be represented by the variation of a vector magnetic potential ψ and the Faraday's law can be mathematically expressed as

$$\vec{E} = -\frac{\partial \vec{\Psi}}{\partial t}, \tag{1.27}$$

by taking curl or cross product of both sides of the equation we have

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{\Psi}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\Psi}) = -\frac{\partial \vec{B}}{\partial t}. \tag{1.28}$$

These relations must hold for any surface bounded by a closed loop, so the last two Maxwell's equations become

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.29}$$

and

$$\vec{\nabla} \times \vec{B} = \mu \left(\vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right). \tag{1.30}$$

Within material media having polarization $\vec{P}$ and magnetization $\vec{M}$ the above laws still hold with the following replacements

$$\vec{\rho} \Rightarrow \vec{\rho} - \vec{\nabla} \times \vec{P}, \qquad \vec{j} \Rightarrow \mu \left(\vec{j} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right). \tag{1.31}$$

That is to the true charge density we have to add the polarization charge density and to the true current density we have to add the contributions of the magnetization current, the polarization current and the displacement current introduced by Maxwell. In terms of the electric displacement and magnetic fields, defined defined by $\vec{D} = \varepsilon \vec{E} + \vec{P}$ and $\vec{H} = \frac{1}{\mu} \vec{B} - \vec{M}$ respectively, Maxwell equations can be brought into the following form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad M1$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad M2$$

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad M3$$

$$\vec{\nabla} \cdot \vec{B} = 0\ . \qquad \text{M4}$$

The set of equations Maxwell's equations expressed in terms of the derived field quantities $\vec{D}$

and $\vec{H}$ are called *Maxwell's macroscopic equations*. These equations are convenient to use in certain simple cases. Together with the boundary conditions and the constitutive relations, they describe uniquely (but only approximately!) the properties of the electric and magnetic fields in matter.

In some materials (*Linear media*) it happens that $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ where the quantities ε and μ are called the *dielectric constant* and *magnetic permeability* of the medium respectively.

The Continuity Equation

The electric charge is conserved. Actually we have never observed in the laboratory a violation of this conservation law. This conservation law is expressed by the following *Continuity Equation*

$$\vec{\nabla}.\vec{j} + \frac{\partial \vec{\rho}}{\partial t} = 0 \qquad \text{(M5)}$$

Where j is the charge density and $\vec{j} = \rho \vec{u}$ is the current density. This equation follows from Maxwell equations and it is not an independent hypothesis.

The quantity $\int_S \vec{j}.\vec{dS}$ represents the charge flowing out of surface S per unit time (this is measured in Amperes in the system MKSA). If the charge density is time independent then from the continuity equation it follows that $\vec{\nabla}.\vec{j} = 0$. In this case we say that we have steady currents.

Remark:

In this system the first three equations (M1)−(M3) are independent, eq. (M4) has the character of an initial condition, and the continuity equation (M5) follows from (M2) and (M3). Indeed, equation (M1) implies that

$$\vec{B}\left(\vec{x},t\right) = \vec{B}\left(\vec{x},t_0\right) - \vec{\nabla}\times\int_{t_0}^{t} \vec{E}(\vec{x},\tau)d\tau ,$$

where

$$\vec{\nabla}.\vec{B}\left(\vec{x},t\right) = \vec{\nabla}.\vec{B}\left(\vec{x},t_0\right),$$

so that eq. (M4) holds for all t if it holds at some fixed (say, initial) time t_o. Similarly, the continuity equation (M5) follows by taking the divergence of (M2) and by applying (M3). In its turn, eq. (M3) can be used to eliminate the unknown ρ by defining the electric volume charge density in terms of $\vec{D}$ as

$$\rho := \vec{\nabla}.\vec{D}$$

In this way, the Maxwell system reduces to the two vector equations (M1), (M2) , valid in any material medium, conducting or non-conducting, for the five unknown vector functions $\vec{E}, \vec{D}, \vec{B}, \vec{H}, \vec{J}$ of (x, t). These two vector equations are complemented by three additional vector relations, called constitutive equations, and so the count is right. These constitutive relations are not universally valid but depend upon the properties of the materials under consideration. We can assume to start with that they have the form of local relations

$$\vec{J} = \vec{J}(\vec{E}, \vec{H})$$

$$\vec{D} = \vec{D}(\vec{E}, \vec{H})$$

$$\vec{B} = \vec{B}(\vec{E}, \vec{H})$$

and in fact for many purposes we will take the very simple linear constitutive relations

$$\vec{J} = \gamma \vec{H} \quad \text{(Ohm's law)} \qquad \text{(C1)}$$

$$\vec{D} = \varepsilon \vec{E} \qquad \text{(C2)}$$

$$\vec{B} = \varepsilon \vec{H} \qquad \text{(C3)}$$

where $\gamma = \gamma(x) \geq 0$, is the electric conductivity, γ^{-1} the resistivity, $\varepsilon = \varepsilon(x) \geq \varepsilon_0 > 0$ is the electric permittivity and $\mu = \mu(x) \geq 0$ the magnetic permeability of the material. These relations apply to empty space with $\varepsilon = \varepsilon_0, \mu = \mu_0, \gamma = 0$ and the more common materials can be classified according to the values of the scalar coefficients $\varepsilon, \mu_0, \gamma = 0$ as follows:

$$\begin{cases} \gamma = 0 : \text{dielectrics} \\ 0 < \gamma < \infty : \text{conductors} \\ \gamma = +\infty : \text{perfect conductors} \end{cases}$$

$$\begin{cases} \mu > \mu o : \text{paramagnetic bodies} \\ 0 < \mu < \mu o : \text{diamagnetic bodies} \\ \mu = 0 : \text{superconductors} \end{cases}$$

where ε_0, μ_0 are the (constant) permittivity and permeability of empty space.
We will exclude in the sequel the case of superconductors [8] and we will always assume that there exists $\bar{\mu} > 0$ such that $\mu(x) = \bar{\mu} > 0$.
For homogeneous media the coefficients γ, ε and μ are constant. They depend on physical parameters such as temperature: for example, the conductivity of metals decreases with increasing temperature.

Electromagnetic Wave Equation and Theory of Lights

A common question is, how are Maxwell's equations used to show wave motion? Consider the electric and magnetic fields in a charge free vacuum region. Then Maxwell's equations become

$$
\begin{aligned}
&\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\
&\vec{\nabla}\times\vec{B} = \mu_0\,\vec{j}(x,t) + \mu_0\varepsilon_0\,\frac{\partial\vec{E}}{\partial t} \\
&\vec{\nabla}\cdot\vec{E} = \rho \\
&\vec{\nabla}\cdot\vec{B} = 0
\end{aligned}
\quad .
$$

Wave Equation for $\vec{E}$

To derive the wave equation for the electric field, start with the third of Maxwell's equations and take the curl of both sides

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})\,. \tag{1.32}$$

The left hand side can be simplified by using the vector relationship

$$\vec{a}\times\vec{b}\times\vec{c} = \vec{b}\left(\vec{a}\cdot\vec{c}\right) - \vec{c}\cdot\left(\vec{a}\cdot\vec{b}\right) \tag{1.33}$$

to get

$$
\begin{aligned}
\vec{\nabla}\times\vec{\nabla}\times\vec{E} &= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{E} \\
&= -\nabla^2\vec{E}
\end{aligned}
\,, \tag{1.34}
$$

where the last step used the fact that $\vec{\nabla}\cdot\vec{E} = 0$. To evaluate the right hand side of (1.32), we start with the fact that the spatial derivatives (∇) and the time derivative can be interchanged. We then use the last of Maxwell's equations to find

$$
\begin{aligned}
-\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} &= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) \\
&= -\frac{\partial}{\partial t}\left(\mu_0\,\vec{j}(x,t)+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \\
&= -\mu_0\frac{\partial}{\partial t}\left(\vec{j}(x,t)+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \quad . \qquad (1.35) \\
&= -\mu_0\frac{\partial}{\partial t}\left(\sigma\vec{E}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)
\end{aligned}
$$

Combining (1.34) and (1.35), (1.32) on rearrangement can be written as

$$
\begin{aligned}
& -\nabla^2\vec{E} = -\mu_0\sigma\frac{\partial\vec{E}}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}, \\
\Rightarrow & \qquad\qquad\qquad\qquad\qquad (1.36) \\
& \nabla^2\vec{E}-\mu_0\sigma\frac{\partial\vec{E}}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0,
\end{aligned}
$$

which we recognize as the three dimensional wave equation for each component of the electric field ($\vec{E}$). Comparing (1.36) with the standard result for a wave whose velocity is v, we obtain

$$
\begin{aligned}
v &= \frac{1}{\sqrt{\mu_0\varepsilon_0}} \\
&= \frac{1}{\sqrt{\left(4\pi\times10^{-7}\ \frac{\text{m}\cdot\text{kg}}{\text{C}^2}\right)\left(8.85\times10^{-12}\ \frac{C^2}{J\cdot m}\right)}} . \qquad (1.37) \\
&= 3.00\times10^{8}\ \tfrac{\text{m}}{\text{s}}.
\end{aligned}
$$

Using the fact that the experimentally determined speed of light is also 3.00 x 10^8 m/s, we are lead to the inescapable conclusion that light is just one form of electromagnetic wave propagation. When the electromagnetic disturbance is moving in a vacuum, we denote its speed by a special symbol, *c*.

Wave Equation for $\vec{B}$

In a manner similar to those leading to eqn. (1.36), we can start with the last of Maxwell's equations to find the wave equation for the magnetic field. Thus,

$$
\begin{aligned}
\vec{\nabla}\times\vec{\nabla}\times\vec{B} &= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{B} \\
&= -\nabla^2\vec{B} \\
&= \vec{\nabla}\times\left(\mu_0\,\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \\
&= \mu_0\sigma\left(\vec{\nabla}\times\vec{E}\right)+\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right). \\
&= -\mu_0\sigma\left(\frac{\partial\vec{B}}{\partial t}\right)-\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{B}}{\partial t}\right).
\end{aligned}
\tag{1.38}
$$

Finally, we have

$$
\begin{aligned}
\nabla^2\vec{B}-\mu_0\sigma\left(\frac{\partial\vec{B}}{\partial t}\right)-\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} &= 0, \\
\nabla^2\vec{B}-\mu_0\sigma\left(\frac{\partial\vec{B}}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} &= 0.
\end{aligned}
\tag{1.39}
$$

This is the wave equation for the magnetic field. We notice that it is exactly the same form as the wave equation for the electric field eqn. (1.36).

Light as Transverse Waves

We can also determine whether light waves are longitudinal or transverse waves. Remember that longitudinal waves oscillate in the same direction as the direction of propagation, while transverse waves oscillate in a direction perpendicular to the direction of propagation. For simplicity, let the direction of propagation be in the x direction. Then $\vec{E} = \vec{E}\ (x,t)$. Now look at a Gaussian box oriented along the coordinate axes. The flux is through the faces in the y-z planes, so Gauss's law becomes

$$\frac{\partial E_x}{\partial x} = 0.$$

From this, we see that the electromagnetic wave has no electric field component in the direction of propagation. Thus, the electric field is exclusively transverse. A similar argument can be used on Gauss's law for magnetic fields to show that it is also transverse to the direction of propagation. In particular, Faraday's law tells us that

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \tag{1.40}$$

In other words, the time dependent magnetic field can only have a component in the z direction when the electric field is exclusively in the y direction. From these, we see that, **in free space, the plane electromagnetic wave is transverse**.

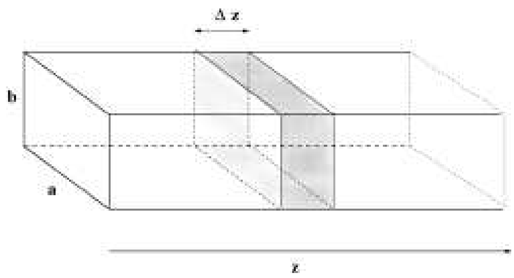

Figure 5**: A rectangular wave guide.**

Plane electromagnetic waves in non-conducting media($\sigma = 0$)

In a medium with values ε, μ for the electric constant and the magnetic permeability respectively, we have derived the Maxwell laws in (1.32) and (1.39) as

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t},$$
$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \sigma \left(\frac{\partial \vec{B}}{\partial t} \right). \qquad (1.41)$$

In region where there are no charge and current distributions, the terms on the right hand sides of (1.41) are absent and the fields $\vec{E}$ and $\vec{B}$ satisfy the free wave equations.

The plane waves are particular solutions of (1.41) in regions where sources are absent. In the following we shall use complex notation and write the electric component of a plane wave as

$$\vec{E} = \vec{E}_0 \exp \ i(\vec{k} . \vec{x} - \omega t). \qquad (1.42)$$

The physical electric field measured in the laboratory is meant to be the real part of this expression. That is $\vec{E} = \vec{E}_0 \cos(\vec{k} . \vec{x} - \omega t)$. A similar expression holds for the magnetic field too with $\vec{E}, \vec{E}_0$ replaced with $\vec{B}, \vec{B}_0$ respectively. In this expression $\vec{E}_0$ is the amplitude of the electric field, $\vec{k}$ is its *wave vector* and ω its *frequency*. This monochromatic pulse is a solution when the frequency is linearly related to the magnitude $k \equiv |\vec{k}|$ of the wave vector $\vec{k}$, by the relationship $\omega = vk$

k is called the *wave number* and is related to the *wave length* by the relation $k = \frac{2\pi}{\lambda}$.

Using Gauss's, law $\vec{\nabla} \cdot \vec{E} = 0$, and the Faraday's law, $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, one can immediately arrive at the following relations for the wave number and the amplitudes of the electric and magnetic components:

$$\vec{k} . \vec{E}_0 = 0, , \quad \vec{B}_0 = \frac{1}{\omega} \vec{k} \times \vec{E}_0 . \qquad (1.43)$$

Eqs. (1.35) state that the electric and magnetic fields of a plane wave are perpendicular to each other and both perpendicular to the direction of the propagation $\vec{n} = \frac{1}{k} \vec{k}$.

Plane electromagnetic waves in conducting media($\sigma \neq 0$)

Within a conductor the electric current density and the electric field are related by $\vec{j} = \sigma \vec{E}$, from which it follows that $\vec{\nabla} \cdot \vec{j} = \sigma \vec{\nabla} \cdot \vec{E} = \frac{\sigma}{\varepsilon}\rho$. Then from the continuity equation one has

$$\vec{\nabla} . \vec{j} + \frac{\partial \vec{\rho}}{\partial t} = \frac{\partial \vec{\rho}}{\partial t} + \frac{\sigma}{\varepsilon}\rho = 0 \,,$$

whch is immediately solved to yield

$$\rho(\vec{x},t) = \rho(\vec{x},0)\exp\left(-\frac{\sigma}{\varepsilon}t\right) . \qquad (1.44)$$

For good conductors $\frac{\sigma}{\varepsilon} \approx 10^{14}\sec^{-1}$ so that from the eqn. (1.36) we conclude that *charges move almost instantly to the surface of the conductor*. The ratio $\tau = \frac{\varepsilon}{\sigma}$ is called the *relaxation time* of the conducting medium. For perfect conductors, $\sigma = \infty$, so that the relaxation time is vanishing. For good, but not perfect, conductors τ is small and of the order of $10^{-14}\sec$ or so. For times much larger than the relaxation time there are practically no charges inside the conductor. All of them have moved to its surface where they form a charge density $\sum$. Within a conductor the wave equation for the vector field $\vec{E}$, see eqn. (1.41), becomes

$$\left(\nabla^2 - \mu\,\varepsilon\,\frac{\partial^2}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\right)\vec{E} = 0 \,.$$

Notice the appearance of a "friction" term $\frac{\partial}{\partial t}$ which was absent in the free wave equation. If we seek for monochromatic solutions of the form $\vec{E} = \vec{E}(\vec{x})\exp(-i\omega t)$, then the equation above takes on the form

$$\left(\nabla^2 + K^2\right)\vec{E}(\vec{x}) = 0 \,,$$

where $K^2 = \mu\omega(\omega\varepsilon + i\sigma)$. This can be immediately solved to yield, for a plane wave solution travelling along an arbitrary direction $\vec{n}$,

$$\vec{E} = \vec{E}(\vec{x})\exp\{i(\alpha\xi - \omega t) - \beta t\} \qquad (1.45)$$

where $\xi = \vec{n} . \vec{x}$. The constants α, β, appearing in (1.45), have dimensions of $length^{-1}$ and are functions of σ . Their analytic expresions are not presented here. These can be traced in any standard book of Electromagnetic Theory. However we can distinguish two particular cases in which their forms are simplified a great deal. These regard the case of an isolator and the case of a very good conductor respectively. For an isolator $\sigma = 0$ and $\alpha = k, \beta = 0$ In this case (1.45) reduces to an ordinary plane wave which is propagating with wave vector $\vec{k} = \vec{n} k$.

For a very good conductor, and certainly this includes the case of a perfect conductor, the conductivity is large so that the range of frequencies with $\sigma >> \varepsilon\omega$ is quite broad. In this case the constants α, β are given by $\alpha \approx \beta \approx \delta^{-1}$, where δ is a constant called the *Skin Depth* , given by the following expression

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{1.46}$$

Therefore we see from eq. (1.45) that inside a good conductor :
The field is attenuated in the direction of the propagation and its magnitude decreases exponentially $\sim \exp\left(-\frac{\xi}{\delta}\right)$ as it penetrates into the conductor. The depth of the penetration is set by δÆ and is smaller the higher the conductivity, the higher the permeability and the frequency. As an example for copper $\sigma = 5.8\times10^{7}$ mho m^{-1} and the skin depth is $\delta \approx 0.7\times10^{-3} cm$ for a frequency $\omega = 100 MHz$. It is important to point out that the magnetic field within the conductor is related to the electric field by the relation

$$\vec{H} = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\sigma}{\mu\omega}}\,\vec{n}\times\vec{E} \ . \tag{1.47}$$

As in the case of non-conducting materials both $\vec{E}$, $\vec{H}$ are perpendicular to each other and to the direction of propagation $\vec{n}$. From (1.47), it is evident that the magnetic field has a phase difference of 45^{0} from its corresponding electric component $\vec{E}$, due to the prefactor $1+i$.

2. Reflection and refraction of plane boundary of Plane Waves.

Introduction

In reality, plane electromagnetic waves frequently encounters obstacles along their propagation paths: hills, buildings, metallic antennas aimed at receiving the messages the waves carry,objects from which they are supposed to partly reflect. In such cases, the wave induces conduction currents in the object(if the object is metallic), or polarization current(if the object is made of an insulator). These current are, of course, sources of a secondary electromagnetic field. This field is known as scattered field, and the process that creates it is known as scattering of electromagnetic waves. The objects, or obstacles are called scatters.

When plane waves are incident on a boundary between different media, some energy crosses the boundary, and some is reflected. In other words, when a plane electromagnetic is incident on a planar boundary between two media, one of these waves is radiated back into the half-space of the incident wave: this wave is known as the reflected wave. There is also a wave in the other half-space (except in the case of a perfect coductor), propagating generally in a different direction from the incident wave;it is therefore called the refracted or transmitted wave. We define transmission and reflection coefficients to quantify the transmission and reflection of wave energy. These coefficients are properties of the two media. The transmission and reflection coefficients are determined by matching the electric and magnetic fields in the waves at the boundary between the two media.

In this session, for easy understanding, we shall consider:
• Boundary conditions on electric and magnetic fields.
• Boundary conditions on fields at the surfaces of conductors.
• Monochromatic plane wave on a boundary:
– directions of reflected and transmitted waves (laws of reflection and refraction);
– amplitudes of reflected and transmitted waves (Fresnel’s equations);
– the special case of a boundary between two dielectrics;
– the special case of the surface of a conductor.
• Monochromatic plane wave on a boundary between two dielectrics:
– polarisation by reflection;
– total internal reflection.
• Reflection coefficient for a conducting surface.

Electromagnetic Boundary Conditions 1: Normal Component of $\vec{B}$

We can use Maxwell’s equations to derive the boundary conditions on the magnetic field across a surface. Electromagnetic shows that the normal component of current, electric displacement, and magnetic induction should be continuous when cross a material interface or boundary; while the tangential component of the electric field and the magnetic field should be continuous cross the material interface. Let us take the magnetic boundary condition as the example to illustrate the calculation. From the Gaussian theorem we have

$$\iiint \vec{\nabla} \cdot \vec{B} dV = \oiint \vec{B} \cdot d\vec{S}.$$

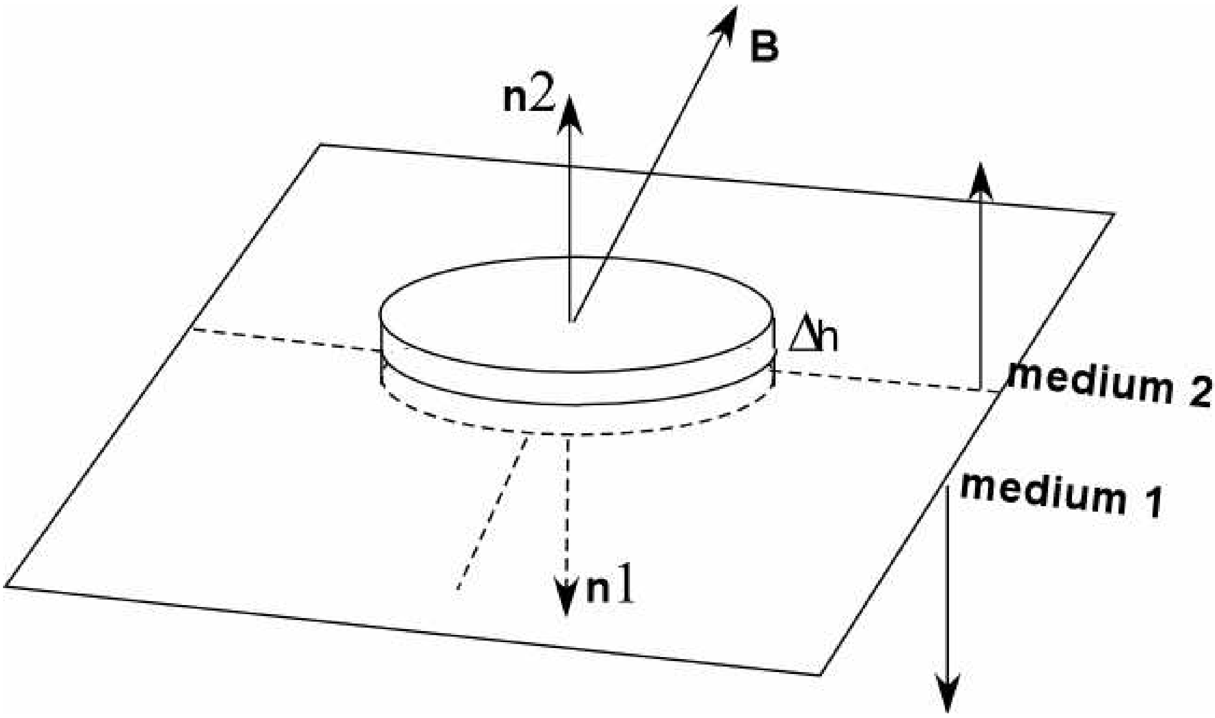

Figure 6. Illustration of the electromagnetic boundary conditions.

By making a small disc with the thickness of Δh and its central line is coincident with the boundary of two media (Figure 6) we have

$$\iiint \vec{\nabla} \cdot \vec{B} dV = 0.$$

This coincides with the Maxwell's equation(M4):

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (2.1)$$

integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \vec{\nabla} \cdot \vec{B} dV = \oint_S \vec{B}.\vec{dS} = 0 \qquad (2.2)$$

where V is the volume of the pillbox, and S is its surface. We can break the integral over the surface into three parts: over the flat ends (S1 and S2) and over the curved wall (S3):

$$\int_{S_1} \vec{B}.\vec{dS} + \int_{S_2} \vec{B}.\vec{dS} + \int_{S_3} \vec{B}.\vec{dS} = 0 \ . \qquad (2.3)$$

In the limit that the length of the pillbox approaches zero, the integral over the curved surface also approaches zero. If each end has a small area A, then equation (2.3) becomes:

$$B_{1n} A + B_{2n} A = 0 \qquad (2.4)$$

or

$$B_{1n} = B_{2n} \qquad (2.5)$$

In other words, the normal component of the magnetic field $\vec{B}$ must be continuous across the surface. By using similar approaches, the general conditions on electric and magnetic fields at the boundary between two materials can be summarised as follows:

Boundary Condition	Derived from…	Applied to…
$B_{1n} = B_{2n}$	$\vec{\nabla} \cdot \vec{B} = 0$	pillbox
$E_{2t} = E_{1t}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Loop
$D_{2n} - D_{1n} = \rho$	$\vec{\nabla} \cdot \vec{D} = \rho$	Pillbox
$H_{2t} - H_{1t} = -J$	$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{D}$	Loop

Static electric fields cannot persist inside a conductor. This is simply because the free charges within the conductor will re-arrange themselves to cancel any electric field; this can result in a surface charge density, ρ. We have seen that electromagnetic waves can pass into a conductor, but the field amplitudes fall exponentially with decay length given by the skin depth, δ:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \qquad (2.6)$$

As the conductivity increases, the skin depth gets smaller. Since both static and oscillating electric fields vanish within a good conductor, we can write the boundary conditions at the surface of such a conductor:

$$E_{1t} \approx 0, E_{2t} \approx 0$$
$$D_{1n} \approx -\rho, D_{2n} \approx 0$$

Lenz's law states that a changing magnetic field will induce currents in a conductor that will act to oppose the change. In other words, currents are induced that will tend to cancel the magnetic field in the conductor. This means that a good conductor will tend to exclude magnetic fields. Thus the boundary conditions on oscillating magnetic fields at the surface of a good conductor can be written:

$$B_{1n} \approx 0, B_{2n} \approx 0$$
$$H_{1t} \approx -J, H_{2t} \approx 0.$$

We can consider an "ideal" conductor as having infinite conductivity. In that case, we would expect the boundary conditions to become:

$$B_{1n} = 0, B_{2n} = 0$$
$$E_{1t} = 0, E_{2t} = 0$$
$$D_{1n} = -\rho, D_{2n} = 0$$
$$H_{1t} = J, H_{2t} = 0.$$

Strictly speaking, the boundary conditions on the magnetic field apply only to oscillating fields, and not to static fields.But it turns out that for superconductors, static magnetic fields are excluded as well as oscillating magnetic fields. This is not expected for classical "ideal" conductors.

Waves on Boundaries

We now apply the boundary conditions to an electromagnetic wave incident on a boundary between two different materials. We shall use the boundary conditions to derive the properties of the reflected and transmitted waves, for a given incident wave. Consider a monochromatic wave incident at some angle on a boundary. We must consider three waves: the incident wave itself; the reflected wave, and the transmitted wave on the far side of the boundary.
The electric field components for these waves can be written (respectively):

$$\vec{E}_I(\vec{r},t) = \vec{E}_{0I}\, e^{i(\omega_I t - \vec{k}_I . \vec{r})} \tag{2.7}$$

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R}\, e^{i(\omega_R t - \vec{k}_R . \vec{r})} \tag{2.8}$$

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T}\, e^{i(\omega_T t - \vec{k}_T . \vec{r})} \tag{2.9}$$

Let us first consider the time dependence of the waves. The boundary conditions must apply at all times: for example, the tangential component of the electric field, $\vec{E}_t$ must be continuous across the boundary at all points on the boundary at all times. This means that all waves must have the same time dependence, and therefore:

$$\omega_I = \omega_R = \omega_T = \omega \tag{2.10}$$

Reflection at a boundary cannot change the frequency of an incident monochromatic wave. Some surfaces reflect some wavelengths better than others, which is why they can appear coloured under white light; but the frequency of the light does not change.

2.1 Laws of Reflection and Refraction

Now let us consider the relationships between the directions in which the waves are moving. We shall find that these relationships are just the laws of reflection and refraction that we are familiar with from basic optics. However, our goal is now to derive these laws from Maxwell's equations, by applying the boundary condition waves across boundaries. We start from the fact that the boundary conditions must be satisfied at all points on the boundary. This means that the waves must all change phase in the same way as we move from one point to another on the boundary. Since the phase of each of the waves at a position $\vec{r}$ is given by $\vec{k} . \vec{r}$, where $\vec{k}$ is the appropriate wave vector, we must have:

$$\vec{k}_I . \vec{p} = \vec{k}_R . \vec{p} = \vec{k}_T . \vec{p}\,, \tag{2.11}$$

where $\vec{p}$ is any point on the boundary.

Laws of Reflection and Refraction

For simplicity, let us choose our coordinates so that the boundary lies in the plane z = 0. Then any point $\vec{p}$ on the boundary can be written:

$$\vec{p} = (x, y, 0) \tag{2.12}$$

Now we can (without loss of generality) further specify the coordinate system so that $\vec{k}_I$ lies in the x − z plane, i.e. the y component of $\vec{k}_I$ is zero:

$$\vec{k}_I = (k_I \sin\theta_{I,} 0, k_I \cos\theta_I) \tag{2.13}$$

where θ_I is the angle between the direction of travel of the incident wave and the boundary.

Now let us apply equation (2.11):

$$\vec{k}_I . \vec{p} = \vec{k}_R . \vec{p} = \vec{k}_T . \vec{p}$$

to points on the boundary with x = 0, i.e. $\vec{p} = (0, y, 0)$. We find:

$$k_{Iy} = k_{Ry} = k_{Ty} \ . \tag{2.14}$$

Therefore, the directions of the incident, reflected and transmitted waves all lie in the plane y = 0.

Now let us consider points on the boundary with y = 0, i.e. $\vec{p} = (x, 0, 0)$.
This time, using equation (2.11) gives:

$$k_{Ix} = k_{Rx} = k_{Tx} = k_I \sin\theta_I \tag{2.15}$$

which (since the vertical components of the wave vectors are all zero) can be written:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \ . \tag{2.16}$$

But since the incident and reflected waves are travelling in the same material with the same frequency, the magnitudes of the wave vectors must be the same:

$$k_I = k_R \ . \tag{2.17}$$

Combining equations (2.15) and (2.16) we find:

$$\theta_I = \theta_R \qquad \text{(the law of reflection)} \tag{2.18}$$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \qquad \text{the law of refraction (Snell's law)} \tag{2.19}$$

2.2 Reflection and Refraction at a Boundary Between Dielectrics

As an example, consider a monochromatic wave incident on a boundary between two dielectrics (e.g. air and glass). Since the conductivity is zero on both sides of the boundary, the wave vectors of all waves must be real.
Also, we have:

$$\frac{\omega}{k_I} = v_1 \, , \ \frac{\omega}{k_T} = v_2 \tag{2.20}$$

where v_1 is the phase velocity in medium 1, and v_2 is the phase velocity in medium 2.

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} \qquad (2.21)$$

We define the refractive index n of a material as the ratio of the speed of light in a vacuum to the speed of light in the material:

$$n = \frac{c}{v} \qquad (2.22)$$

Then equation (2.21) can be written:

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \qquad (2.23)$$

This is the familiar form of Snell's law.

Reflection and Refraction at the Surface of a Conductor

For a wave incident on a conductor, $\vec{k}_T$ will be complex:

$$\vec{k}_T = \vec{\alpha} - i\vec{\beta}\ . \qquad (2.24)$$

For a good conductor (i.e. $\sigma >> \omega\varepsilon_2$):

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}}\ . \qquad (2.25)$$

so:

$$k_T = \sqrt{\vec{\alpha}^2 + \vec{\beta}^2} = \sqrt{\omega\mu_2\sigma_2} \qquad (2.26)$$

Applying the law of refraction (2.19):

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \approx \sqrt{\frac{\sigma_2}{\omega\varepsilon_1}} >> 1 \qquad (2.27)$$

where we have assumed that $\mu_2 \approx \mu_1$. Since the largest value of $\sin\theta_I$ is 1, equation (2.27) tells us that $\sin\theta_T \approx 0$, so the direction of the transmitted wave in a good conductor must be (close to the) normal to the surface.

Possible Questions: Waves on Boundaries

1. Derive (from Maxwell's equations) the boundary conditions on electric and magnetic fields at the interface between two media.

2. Apply the boundary conditions on electric and magnetic fields to derive the laws of reflection and refraction.

3. Energy and Momentum

We shall use Maxwells macroscopic equations in(M1,M2,M3,M4) the following considerations on the energy and momentum of the electromagnetic field and its interaction with matter.

3.1 The energy theorem in Maxwell's theory

Scalar multiplying (M1) by $\vec{H}$, (M2) by $\vec{E}$ and subtracting, we obtain

$$\begin{aligned} &\vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{E}) = \nabla\cdot(\vec{E}\times\vec{H}) \\ &= -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{j} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right) - \vec{j}\cdot\vec{E} \end{aligned} \quad . \qquad (6.24)(3.1)$$

Integration over the entire volume V and using Gauss's theorem (the divergence theorem), we obtain

$$-\frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)d^3x' = \int_V \vec{j}\cdot\vec{E}d^3x' + \int_A (\vec{E}\times\vec{H})\cdot\vec{n}\,d^2x' \quad . \qquad (6.25)(3.2)$$

But, according to Ohm's law in the presence of an electromotive force field, the linear relationship between the current and the electric field is

$$\vec{j} = \sigma\left(\vec{E}+\vec{E}^{EMF}\right) \qquad (6.26)(3.3)$$

which means that

$$\int_V \vec{j}\cdot\vec{E}d^3x' = \int_V \frac{j^2}{\sigma}d^3x' - \int_V\left(\vec{j}\cdot\vec{E}^{EMF}\right)d^3x' \qquad (6.27)(3.4)$$

Inserting this into Equation (3.2)

$$\int_V \vec{j}\cdot\vec{E}d^3x' = \int_V \frac{j^2}{\sigma}d^3x' + \frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)d^3x' + \int_A(\vec{E}\times\vec{H})\cdot\vec{n}\,d^2x'$$

(6.28)(3.5)

i.e.

$$\text{Applied electric power} = \text{Joule heat} + \text{Field energy} + \text{Radiated power}$$

which is the ***energy theorem in Maxwell's theory*** also known as ***Poynting's theorem***.

It is convenient to introduce the following quantities:

$$U_e = \frac{1}{2}\int_V \vec{E}\cdot\vec{D}\, d^3x'$$

$$U_m = \frac{1}{2}\int_V \vec{H}\cdot\vec{B}\, d^3x' \qquad (3.6)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

where U_e is the *electric field energy*, U_m is the *magnetic field energy*, both measured in J, and $\vec{S}$ is the *Poynting vector* (*power flux*), measured in W/m^2.

3.2 The momentum theorem in Maxwell's theory

We now investigate the momentum balance (force actions) in the case that a field interacts with matter in a non-relativistic way. For this purpose we consider the force density given by the *Lorentz force* per unit volume $\rho\vec{E}+\vec{j}\times\vec{B}$.

Using Maxwell’s equations (M1-M4) and symmetrising, we obtain

$$\rho\vec{E}+\vec{j}\times\vec{B} = (\nabla\cdot\vec{D})\vec{E}+\left(\nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}\right)\times\vec{B}$$

$$= (\nabla\cdot\vec{D})\vec{E}+(\nabla\times\vec{H})\times\vec{B}-\frac{\partial\vec{D}}{\partial t}\times\vec{B}$$

$$= (\nabla\cdot\vec{D})\vec{E}-\vec{B}\times(\nabla\times\vec{H})-\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)+\vec{D}\times\frac{\partial\vec{B}}{\partial t} \qquad (3.7)$$

$$= (\nabla\cdot\vec{D})\vec{E}-\vec{B}\times(\nabla\times\vec{H})-\frac{\partial}{\partial t}(\vec{D}\times\vec{B})-\vec{D}\times(\nabla\times\vec{E})+\vec{H}(\nabla\cdot\vec{B})$$

$$= \left[(\nabla\cdot\vec{D})\vec{E}-\vec{D}\times(\nabla\times\vec{E})\right]+\left[(\nabla\cdot\vec{B})\vec{H}-\vec{B}\times(\nabla\times\vec{H})\right]$$

$$= -\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)$$

One verifies easily that the *i*th vector components of the two terms in square brackets in the right hand member of (3.7) can be expressed as

$$\left[(\nabla\cdot\vec{D})\vec{E}-\vec{D}\times(\nabla\times\vec{E})\right]_i = \frac{1}{2}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial x_i}-\vec{D}\cdot\frac{\partial\vec{E}}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(E_iD_i-\frac{1}{2}\vec{E}\cdot\vec{D}\,\delta_{ij}\right), \qquad (3.8)$$

and

$$\left[(\nabla \cdot \vec{B})\vec{H} - \vec{B}\times(\nabla\times\vec{H}) \right]_i = \frac{1}{2}\left(\vec{H}\cdot\frac{\partial \vec{B}}{\partial x_i} - \vec{B}\cdot\frac{\partial \vec{H}}{\partial x_i} \right) + \frac{\partial}{\partial x_i}\left(H_i B_i - \frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij} \right) \tag{3.9}$$

respectively.
Using these two expressions in the *i*th component of Equation (3.7) on the preceding page and re-shuffling terms, we get

$$\rho\vec{E} + \vec{j}\times\vec{B} = \frac{1}{2}\left[\left(\vec{E}\cdot\frac{\partial \vec{D}}{\partial x_i} - \vec{D}\cdot\frac{\partial \vec{E}}{\partial x_i} \right) + \left(\vec{H}\cdot\frac{\partial \vec{B}}{\partial x_i} - \vec{B}\cdot\frac{\partial \vec{H}}{\partial x_i} \right) \right] + \frac{\partial}{\partial t}\left(\vec{D}\times\vec{B} \right)$$

$$= \frac{\partial}{\partial x_i}\left(E_i D_i - \frac{1}{2}\vec{E}\cdot\vec{D}\,\delta_{ij} + H_i B_i - \frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij} \right) \tag{3.10}$$

Introducing the *electric volume force* F_{ev} via its *i*th component

$$\left(F_{ev}\right)_i = \left(\rho\vec{E} + \vec{j}\times\vec{B} \right)_i - \frac{1}{2}\left[\left(\vec{E}\cdot\frac{\partial \vec{D}}{\partial x_i} - \vec{D}\cdot\frac{\partial \vec{E}}{\partial x_i} \right) + \left(\vec{H}\cdot\frac{\partial \vec{B}}{\partial x_i} - \vec{B}\cdot\frac{\partial \vec{H}}{\partial x_i} \right) \right] \tag{3.11}$$

and the *Maxwell stress tensor* T with components

$$T_{ij} = E_i D_i - \frac{1}{2}\vec{E}\cdot\vec{D}\,\delta_{ij} + H_i B_i - \frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij} \tag{3.12}$$

we finally obtain the force equation

$$\left[F_{ev} + \frac{\partial}{\partial t}\left(\vec{D}\times\vec{B} \right) \right]_i = \frac{\partial T_{ij}}{\partial x_i} = \left(\nabla\cdot\vec{T} \right)_i \tag{3.13}$$

If we introduce the *relative electric permittivity* k and the *relative magnetic Permeability* k_m as

$$\vec{D} = k\varepsilon_0\,\vec{E} = \varepsilon\,\vec{E} \tag{3.14}$$

$$\vec{B} = k_m\mu_0\,\vec{H} = \mu\,\vec{H} \tag{3.15}$$

we can rewrite (3.13) as

$$\frac{\partial T_{ij}}{\partial x_i} = \left(F_{ev} + \frac{k k_m}{c^2}\frac{\partial \vec{S}}{\partial t} \right)_i \tag{3.16}$$

where $\vec{S}$ is the Poynting vector defined in Equation (3.6). Integration over the entire volume *V* yields

$$\int_V \rho\left(\vec{E}+\vec{v}\times\vec{B}\right)d^3x' + \frac{1}{c^2}\frac{d}{dt}\int_V \vec{S}d^3x' = \int_S \vec{T}_{\hat{n}}d^2x', \tag{3.17}$$

Force on the matter + Field momentum = *Maxwell* Stress

which expresses the balance between the force on the matter, the rate of change of the electromagnetic field momentum and the Maxwell stress. This equation is called the *momentum theorem in Maxwell's theory*. In vacuum (3.17) becomes

$$\int_V \rho\left(\vec{E}+\vec{v}\times\vec{B}\right)d^3x' + \frac{1}{c^2}\frac{d}{dt}\int_V \vec{S}d^3x' = \int_S \vec{T}_{\hat{n}}d^2x', \tag{3.18}$$

Force on the matter + Field momentum = *Maxwell* Stress

or

$$\frac{d}{dt}P^{\mathrm{Mech}} + \frac{d}{dt}P^{\mathrm{Field}} = \int_S \vec{T}_{\hat{n}}d^2x' \quad . \tag{3.19}$$

4. Radiation from Extended Sources

Certain radiation systems have a geometry which is one-dimensional, symmetric or in any other way simple enough that a direct calculation of the radiated fields and energy is possible. This is for instance the case when the current flows in one direction in space only and is limited in extent. An example of this is the linear antenna

4.1 Radiation from charges moving in matter

When electromagnetic radiation is propagating through matter, new phenomena may appear which are (at least classically) not present in vacuum. As mentioned earlier, one can under certain simplifying assumptions include, to some extent, the influence from matter on the electromagnetic fields by introducing new, derived field quantities $\vec{D}$ and $\vec{H}$ according to

$$\vec{D} = \varepsilon(t, \vec{x})\vec{E} = k\varepsilon_0 \vec{E} \tag{4.1}$$

$$\vec{B} = \mu(t, \vec{x})\vec{H} = k_m\mu_0 \vec{H} \,. \tag{4.2}$$

Expressed in terms of these derived field quantities, the Maxwell equations, often called *macroscopic Maxwell equations*, take the form as shown previously[M1-M4] Assuming for simplicity that the *electric permittivity* ε and the *magnetic permeability* μ, and hence the *relative permittivity* k and the *relative permeability* k_m all have fixed values, independent on time and space, for each type of material we consider, we can derive the general *telegrapher's equation*

$$\frac{\partial^2 \vec{E}}{\partial \varsigma^2} - \sigma\mu\frac{\partial \vec{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} = (0,0,0) \tag{4.3}$$

describing (1D) wave propagation in a material medium. It is known that the existence of a finite conductivity, manifesting itself in a *collisional interaction* between the charge carriers, causes the waves to decay exponentially with time and space. Let us therefore assume that in our medium $\sigma = 0$ so that the wave equation simplifies to

$$\frac{\partial^2 \vec{E}}{\partial \varsigma^2} - \varepsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} = (0,0,0) \tag{4.4}$$

If we introduce the *phase velocity* in the medium as

$$v_\varphi = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{kk_m\varepsilon_0\mu_0}} = \frac{c}{\sqrt{kk_m}} \tag{4.5}$$

where, according to Equation (1.29), $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ is the speed of light, *i.e.*, the phase speed of electromagnetic waves in vacuum, then the general solution to each component of Equation (4.4) on the previous page

$$E_i = f(\varsigma - v_\varphi t) + g(\varsigma + v_\varphi t), \qquad i = 1,2,3. \tag{4.6}$$

The ratio of the phase speed in vacuum and in the medium

$$\frac{c}{v_\varphi} = \sqrt{kk_m} = c\sqrt{\varepsilon\mu} \stackrel{\text{def}}{\equiv} n \tag{4.7}$$

is called the *refractive index* of the medium. In general *n* is a function of both time and space as are the quantities ε, μ, k and k_m themselves. If, in addition, the medium is *anisotropic* or *birefringent*, all these quantities are rank-two tensor fields. Under our simplifying assumptions, in each medium we consider *n = Const* for each frequency component of the fields. Associated with the phase speed of a medium for a wave of a given frequency ω we have a *wave vector*, defined as

$$\vec{k} \stackrel{\text{def}}{\equiv} k\hat{\vec{k}} = kv_\varphi = \frac{\omega}{v_\varphi}\frac{\vec{v}_\varphi}{v_\varphi} \tag{4.8}$$

Consider the case of the vacuum where we assume that $\vec{E}$ is time-harmonic, *i.e.*, can be represented by a Fourier component proportional to $\exp\{i\omega t\}$, the solution of Equation (4.4) can be written

$$\vec{\mathrm{E}} = \mathrm{E}_0 \exp\left\{i(\vec{\mathrm{k}}\cdot\vec{\mathrm{x}} - \omega t)\right\} \tag{4.9}$$

where now $\vec{\mathrm{k}}$ is the wave vector *in the medium* given by Equation (4.8). With these definitions, the vacuum formula for the associated magnetic field,

$$\vec{\mathrm{B}} = \sqrt{\varepsilon\mu}\,\hat{\vec{k}}\times\vec{\mathrm{E}} = \frac{1}{v_\varphi}\hat{\vec{k}}\times\vec{\mathrm{E}} = \frac{1}{\omega}\vec{k}\times\vec{\mathrm{E}} \tag{4.10}$$

is valid also in a material medium (assuming, as mentioned, that *n* has a fixed constant scalar value). A consequence of a $k \neq 1$ is that the electric field will, in general, have a longitudinal component. It is important to notice that depending on the electric and magnetic properties of a medium, and, hence, on the value of the refractive index *n*, the phase speed in the medium can be smaller or larger than the speed of light:

$$v_\varphi = \frac{c}{n} = \frac{\omega}{k}\ , \tag{4.11}$$

where, in the last step, we have used eqn. (4.8). If the medium has a refractive index which, as is usually the case, dependent on frequency ω, we say that the medium is *dispersive*. Because in this the *group velocity*

$$v_g = \frac{\partial\omega}{\partial k} \tag{4.12}$$

has a unique value for each frequency component, and is different from *v*'. Except in regions of *anomalous dispersion*, *v*' is always smaller than *c*. In a gas of free charges, such as a *plasma*, the refractive index is given by the expression

$$n^2(\omega) = 1 - \frac{\omega_p{}^2}{\omega^2}\ , \tag{4.13}$$

where

$$\omega_p{}^2 = \sum_\sigma \frac{N_\sigma q_\sigma{}^2}{\varepsilon_0 m_\sigma} \tag{4.14}$$

is the *plasma frequency*. Here m_σ and N_σ denote the mass and number density, respectively, of charged particle species σ. In an inhomogeneous plasma, $N_\sigma = N_\sigma(\vec{x})$ so that the refractive index and also the phase and group velocities are space dependent. As can be easily seen, for each given frequency, the phase and group velocities in a plasma are different from each other. If the frequency ω is such that it coincides with ω_p at some point in the medium, then at that point $v_\varphi \to \infty$ while $v_g \to 0$ and the wave Fourier component at ω is reflected there.

5. Derivation of the Lorentz Transformation

In most cases, the Lorentz transformation is derived from the two postulates: the equivalence of all inertial reference frames and the invariance of the speed of light. However, the most general transformation of space and time coordinates can be derived using only the equivalence of all inertial reference frames and the symmetries of space and time. The general transformation depends on one free parameter with the dimensionality of speed, which can be then identifed with the speed of light *c*. This derivation uses the group property of the Lorentz transformations, which means that a combination of two Lorentz transformations also belongs to the class Lorentz transformations.
The derivation can be compactly written in matrix form. However, for those not familiar with matrix notation, we may also write it without matrices.

5.1:

1) Let us consider two inertial reference frames *O* and *O'*. The reference frame *O'* moves relative to *O* with velocity *v* in along the *x* axis. We know that the coordinates *y* and *z* perpendicular to the velocity are the same in both reference frames: *y* = *y'* and *z* = *z'*. So, it is sufficient to consider only transformation of the coordinates *x* and *t* from the reference frame *O* to $x' = f_x(x; t)$ and $t' = f_t(x; t)$ in the reference frame *O'*. From translational symmetry of space and time, we conclude that the functions $f_x(x; t)$ and $f_t(x; t)$ must be linear functions. Indeed, the relative distances between two events in one reference frame must depend only on the relative distances in another frame:

$$x_1' - x_2' = f_x(x_1 - x_2, t_1 - t_2), \qquad t_1' - t_2' = f_t(x_1 - x_2, t_1 - t_2) \tag{5.1}$$

Because Eq. (5.1) must be valid for any two events, the functions $f_x(x; t)$ and $f_t(x; t)$ must be linear functions. Thus

$$\begin{aligned} x' &= Ax + Bt, \\ t' &= Cx + Dt \end{aligned} \tag{5.2}$$

where A,B,C and D are some coefficients that depend on v. In matrix form Eqns (5.2) are written as

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{5.3}$$

with four unknown function A,B,C and D of v.

2) The origin of the reference frame *O'* has the coordinate *x'* = 0 and moves with velocity *v* relative to the reference frame *O*, so that *x* = *vt*. Substituting these values into Eq. (5.2), we find *B* = -*vA*. Thus, the first equation of Eqs. (5.2) has the form

$$x' = A(x - vt), \tag{5.4}$$

so we need to find only three unknown functions A,C and D of v.
3) The origin of the reference frame *O* has the coordinate *x* = 0 and moves with velocity -*v* relative to the reference frame *O'*, so that *x'* = -*vt'*. Substituting these values in Eqs. (5.2), we find *D* = *A*. Thus, the second part of Eqs. (2) has the form

$$t' = Cx + At = A(Ex + t) \qquad (5.5)$$

where we introduced the new variable E=C/A.

Let us change to the more common notation $A = \gamma$. Then Eqs. (5.4) and (5.5) have the form

$$x' = \gamma(x - vt), \qquad (5.6)$$

$$t' = \gamma(Ex + t) \ , \qquad (5.7)$$

or in matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ E & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \qquad (5.8)$$

Now we need to find only two unknown functions γ_v and E_v of v.

4) A combination of two Lorentz transformations also must be a Lorentz transformation.
Let us consider a reference frame *O'* moving relative to *O* with velocity v_1 and a reference frame *O''* moving relative to *O'* with velocity v_2. Then

$$\begin{aligned} x'' &= \gamma_{v2}(x' - v_2 t'), & x' &= \gamma_{v1}(x - v_1 t), \\ t'' &= \gamma_{v2}(E_{v2} x' + t'), & t' &= \gamma_{v1}(E_{v1} x + t), \end{aligned} \qquad (5.9)$$

which can also be put in the matrix form as done earlier.

For a general Lorentz transformation, the coefficients in front of *x* in Eq. (5.6) and in front of *t* in Eq. (5.7) are equal, i.e. the diagonal matrix elements in Eq. (8) are equal.

If we substitute for x' and t' in the first equation of Eqs. (5.9), we obtain

$$\begin{aligned} x'' &= \gamma_{v2}\gamma_{v1}\left[(1 - E_{v1} v_2)x - (v_1 + v_2)t\right] \\ t'' &= \gamma_{v2}\gamma_{v1}\left[(E_{v1} + E_{v1})x + (1 - E_{v1} v_2)t\right]. \end{aligned} \qquad (5.10)$$

Similarly, Eq. (5.10) must also satisfy this requirement:

$$1 - E_{v1} v_2 = 1 - E_{v2} v_1 \quad \Rightarrow \quad \frac{v_2}{E_{v2}} = \frac{v_1}{E_{v1}} \qquad (5.11)$$

In the second Eq. (5.14), the left-land side depends only on v_2, and the right-hand side only on v_1. This equation can be satisfied only if the ratio v/E_v is a constant *a* independent of velocity *v*, i.e.

$$E_v = \frac{v}{a}, \qquad (5.12)$$

Substituting Eq. (5.12) into Eqs. (5.6) and (5.7), as well as (5.8), we find

$$x' = \gamma_v (x - vt), \quad t' = \gamma_v \left(v/a \, x + t \right) \ , \qquad (5.13)$$

or in the matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & -v \\ \frac{v}{a} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \qquad (5.14)$$

Now we need to find only one unknown function γ_v, whereas the coefficient *a* is a fundamental constant independent on *v*.

5) Let us make the Lorentz transformation from the reference frame *O* to *O'* and then from *O'* back to *O*. The first transformation is performed with velocity *v*, and the second transformation with velocity -*v*. The equations are similar to Eqs. (5.10):

$$x = \gamma_{-v}\left(x' + v\,t'\right), \qquad x' = \gamma_v\left(x - v_1 t\right),$$
$$t = \gamma_{-v}\left(\frac{v}{a}x' + t'\right), \qquad t' = \gamma_v\left(\frac{v}{a}x + t\right), \quad . \qquad (5.15)$$

Substituting *x'* and *t'* from the first equation (5.15) into the second one, we find

$$x = \gamma_{-v}\gamma_v\left(1 + {v^2}/{a}\right)x, \quad t = \gamma_{-v}\gamma_v\left(1 + {v^2}/{a}\right). \qquad (5.16)$$

Eqn.(5.15) must be valid for any x and t, so

$$\gamma_{-v}\gamma_v = \frac{1}{1 + \dfrac{v^2}{a}} \qquad (5.17)$$

Because of the space symmetry, the function γ_v must depend only on the absolute value of velocity *v*, but not on its direction, so $\gamma_{-v} = \gamma_v$. Thus we find

$$\gamma_v = \frac{1}{\sqrt{1 + \dfrac{v^2}{a}}}\,. \qquad (5.18)$$

6) Substituting Eq. (5.18) into Eqs. (5.13) and (5.14) , we find the final expressions for the transformation as

$$x' = \frac{x - vt}{\sqrt{1 + \dfrac{v^2}{a}}}, \qquad t' = \frac{{v}/{a}\,x + t}{\sqrt{1 + \dfrac{v^2}{a}}}, \qquad (5.19)$$

which can also be put in the matrix form.
Eqs. (5.19) and its matrix equivalent have one fundamental parameter *a*, which has the dimensionality of velocity squared. If *a* < 0, we can write it as

$$a = -c^2\,. \qquad (5.20)$$

Then Eqs. (5.19) and its matrix eqivalent become the standard Lorentz transformation:

$$x' = \frac{x - vt}{\sqrt{1 - \dfrac{v^2}{a}}}, \qquad t' = \frac{{v}/{a}\,x + t}{\sqrt{1 - \dfrac{v^2}{a}}}, \qquad (5.21)$$

It is easy to check from Eq. (5.21) that, if a particle moves with velocity *c* in one reference frame, it also moves with velocity *c* in any other reference frame, i.e. if *x* = *ct* then *x'* = *ct'*. Thus the parameter *c* is the invariant speed. Knowing about the Maxwell equations and electromagnetic waves, we can identify this parameter with the speed of light. It is straightforward to check that the Lorentz transformation (5.21) and matrix equivalent preserves the space -time interval

$$\left(ct'\right)^2 - x'^2 = \left(ct\right)^2 - x^2, \qquad (5.22)$$

or it has the Minkowski metric.

If $a = \infty$, then Eqs. (5.22) and (5.21) produce the non-relativistic Galileo transformation:

$$x' = \gamma(x - vt), \qquad t' = t \tag{5.23}$$

or in matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}.$$

If $a > 0$, we can write it as $a = \sigma^2$. Then Eqs. (5.19) describes a Euclidean space-time and preserve the space-time distance:

$$(\sigma t')^2 + x'^2 = (\sigma t)^2 + x^2 \tag{5.24}$$

Problem 1:

(A) By examination of (23) show that

$$\omega = \frac{2\pi}{T} \qquad (26)$$

where T is the time for the electric field to complete one cycle at a fixed z. That is, where T is the temporal period.

(B) By examination of (23) show that

$$k == \frac{2\pi}{\lambda} \qquad (27)$$

where λ is the wavelength (that is, the spatial period) of the wave.

(C) By examination of (23) show that the phase velocity of this wave is indeed ω / k.

Problem 2:

Use Eq. (17) above to show that the magnetic field $\vec{b}$ corresponding to the electric field (23) has the form,

$$\vec{b} = \frac{1}{c} E_0 \cos(\omega t - kz + \phi_0)\hat{y} \qquad (28)$$

To do this problem, substitute (23) into the right hand side of (17) and then integrate to find the magnetic field. (**Do not just show that (28) works upon substitution.)** In doing the integration be careful to keep track of the limits.