



**NATIONAL OPEN UNIVERSITY OF NIGERIA**

**SCHOOL OF SCIENCE AND TECHNOLOGY**

**COURSE CODE: MTH 382**

**COURSE TITLE: MATHEMATICAL METHODS IV**

**MTH 382: MATHEMATICAL METHODS IV**

Course Team: Dr. Bankole Abiola (Writer) – NOUN  
Dr. Bankole Abiola (Editor) – NOUN  
Dr. Bankole Abiola (Programme Leader) – NOUN  
Dr S.O. Ajibola  
National Open University of Nigeria  
14-16 Ahmadu Bello Way, Victoria Island Lagos.

**MODULE 1: EXISTENCE AND UNIQUENESS OF SOLUTIONS**

UNIT1: Ordinary Differential Equations

UNIT2: The Fixed Point Method

UNIT3: The Method of Successive Approximation

**MODULE 2: SPECIAL FUNCTIONS**

UNIT1: Special Functions

UNIT2: Hyper geometric Function

UNIT3: Bessel Functions

**MODULE 3: SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATION**

UNIT1: Legendry Function

UNIT2: Some Examples of Partial Differential Equation

## MODULE 1: EXISTENCE AND UNIQUENESS OF SOLUTION

### UNIT 1: Ordinary Differential Equation

1.0. Introduction

2.0. Objectives

3.0. Main Content

3.1. Definitions and Examples

4.0. Conclusion

5.0. Summary

6.0 Tutor Marked Assignment

7.0. References/Further Readings

1.0. Introduction: In this module we shall start the theory of ordinary differential equations with a discussion on existence and uniqueness theorems covering some various types of equations using various approaches. A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to a single variable in the case of an ordinary differential equation. The order of the highest derivative is called the order of the equation. Derivatives in a differential equation can occur in various ways and we do not admit equations where the unknown is subjected to other operations than algebraic and differential equations.

2.0. Objectives: At the end of this unit you should be able

- to classify various types of differential equation
- to answer correctly exercises on differential equations

### 3.0

### MAIN CONTENT

#### 3.1. Definitions and Examples

Definition (1): A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to single variables in the case of an ordinary differential equation.

Examples: Consider the following list of six examples of functional equations involving derivatives contains some bona fide equations and some which are not.

Example (1):  $f'(x) = f(x)$

Example (2):  $f'(x) = f(x+1)$

Example (3):  $f'(x) = a_0(x) + a_1(x)f(x) + a_2(x)[f(x)]^2$

Example (4):  $f''(x) = 6x + [f(x)]^2$

Example (5):  $f'(x) = \int_0^x \{1 + [f(s)]^2\}^{1/2} ds$

Example (6):  $f(x) = \int_0^1 \{[f'(s)]^2 + [f(x)]^2\}^{1/2} ds$

Examples 1 and 3 are ordinary and first order differential equations, while Example 2 is a difference differential equation, not a differential equation in the usual sense. Example 4 is a second order differential equation. Example 5 is not a differential equation as it stands but on differentiating will yield

$f''(x) = \{1 + [f(x)]^2\}^{1/2}$  which is a second order differential equation which is equivalent to Example 4. Finally Example 6 is not a differential equation and is not reducible to such an equation by elementary means.

The normal form of a first order differential equation is given as

$$y' = F(x, y) \dots (1)$$

In the simplest case  $x$  and  $y$  are real variables and  $F(x, y)$  is a function on  $R^2$  to  $R^1$ . We can also allow  $x$  and  $y$  to be complex variables and  $F$  to be a function on  $C^2$  to  $C^1$

We can also let

$$y = (y_1, y_2, y_3, y_4, \dots, y_n) \text{ and } F = (F_1, F_2, F_3, \dots, F_n) \dots (2)$$

Where  $y$  and  $F$  are functions on  $R^{n+1}$  to  $R^1$  We then define the derivative of a vector as the vector of the derivatives:

$$y' = (y'_1, y'_2, y'_3, \dots, y'_n) \quad \dots \quad (3)$$

With this notation equation (1) becomes a condensed convenient way of writing a system of first order differential equations:

$$y'_j(x) = F_j(x, y_1, y_2, y_3, \dots, y_n), j = 1, 2, 3, \dots, n \quad \dots \quad (4)$$

Conversely every such system can be writing as a first order vector differential equation. The generalization has the advantage of covering  $n$ th order equations.

To convert  $n$ th order differential equation in  $y$  to a first order vector equation in  $y$ , we set

$$y = (y, y', y'', \dots, y^{(n-1)}) \quad \dots \quad (5)$$

We can consider differential equations in more general spaces than the Euclidean. Here the interpretation of the derivatives becomes a matter of concern, and convergence questions also arise if the space is of infinite dimension.

Differential equations normally have infinite number of solutions and in order to find a particular one we have to impose some special condition on the solution, usually an initial condition. The intent of an existence theorem is to show that there exists a function which satisfies the equation in some neighborhood of point  $(x_0, y_0)$ . A Uniqueness theorem asserts that there is only one such function. We can, however assert the existence of solution under much more general conditions than those which guarantee uniqueness. This is beyond the scope of this course.

4.0. Conclusion: We have examined differential equations in a general setting in this unit. This unit is important to the understanding of other materials that would follow subsequently.

5.0 Summary: In this unit we have a general introduction to various form of differential equations. This unit must be read carefully before proceeding to the next units

6.0. Tutor Marked Assignments.

(1) If  $f(x)$  satisfies the integral equation

$$f(x) = y_0 + \int_{x_0}^{x_1} F[s, f(s)]ds,$$

Find a differential satisfied by  $f(x)$ . What initial condition does  $f(x)$  satisfy?

(2). Transform  $f(x) = \int_0^x [f(s)]^2 ds$  into differential equation. Here  $f(x) = 0$

is obviously a solution. Are there other solutions of the functional equation?

(3). The functional equation  $f(x) = 1 + \int_0^x f(s)ds$  implies that  $f$  satisfies a differential equation. Find the latter and find the common solution.

#### 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London

## UNIT2: The Fixed Point Method

- 1.0. Introduction
- 2.0. Objectives
- 3.0. Main Content
  - 3.1. The Fixed Point Method
- 4.0. Conclusion
- 5.0. Summary
- 6.0 Tutor Marked Assignment
- 7.0. References/Further Readings

1.0 Introduction: In this unit we shall use a topological method based on the contraction fixed point theorem. To apply this theorem successfully we have to replace the differential equation by an equivalent integral equation that can be used to define a contraction operator on a suitably chosen metric space.

2.0. Objectives: At the end of this unit you should be able to apply the contraction fixed point theorem to

- determine the existence of solutions for a given differential equation

- solve correctly the tutor marked assignments that follows.

### 3.0 MAIN CONTENT

#### 3.1 The Fixed Point Method

Consider the following differential equation defined by

$$f'(x) = F[x, f(x)], f(x_0) = y_0 \quad \dots (1)$$

Here  $F = (F_1, F_2, \dots, F_n)$  is a vector valued function defined and continuous in  $B : |x - x_0| < a, \|y - y_0\| < b$

We may define the norm on  $R^n$  as follows:

$$\left[ \sum_1^n (y_j - y_{j0}) \right]^{1/2} \quad \text{or} \quad \max |y_j - y_{j0}|$$

We impose two further conditions on  $F$  :

$$\|F(x, y)\| \leq M \quad \dots (2)$$

$$\|F(x, y_1) - F(x, y_2)\| \leq K \|y_1 - y_2\| \quad \dots (3)$$

Condition (1) and (2) are called boundedness and Lipschitz condition respectively.

We now replace the vector differential equation by a vector integral equation defined as:

$$f(x) = y_0 + \int_{x_0}^x F[s, f(s)] ds \quad \dots (4)$$

We again impose the following property which follows from the definitions of integrals by Riemann sums, defined as:

$$\left\| \int_{x_0}^x F ds \right\| \leq \int_{x_0}^x \|F\| ds, x_0 < x \quad \dots (5)$$

Theorem (1): Under the stated assumptions on F, the equation (1) has a unique solution defined in the interval  $(x_0 - r, x_0 + r)$  where

$$r < \min\left(a, \frac{b}{M}, \frac{1}{K}\right)$$

Proof: We consider the space N of all functions  $g(x)$  on  $R^1$  to  $R^n$  continuous in  $x$   $(x_0 - r, x_0 + r)$  such that  $g(x_0) = y_0$  and  $\|g - y_0\|_N \leq b$  where  $\|g - y_0\|_N = \sup_x \|g(x) - y_0\|$ . For such a  $g(x)$  the function  $F[x, g(x)]$  exists and is continuous. Further its N-norm does not exceed M. We now define the transformation:

$$T : g(x) \rightarrow y_0 + \int_{x_0}^x F[s, g(s)]ds, \quad -r < x - x_0 < r \quad \dots (6)$$

Here  $T[g](x)$  is continuous,  $T[g](x_0) = y_0$  and

$$\|T[g](x) - y_0\| < Mr < b \quad \dots (7)$$

(by the choice of r). It follows that  $T[g] \in N$ . We next observe that

$$\|T[g_1](x) - T[g_2](x)\| = \left\| \int_{x_0}^x \{F[s, g_1(s)] - F[s, g_2(s)]\} ds \right\| < K \left\| \int_{x_0}^x \|g_1(s) - g_2(s)\| ds \right\|$$

This shows that

$$\|T[g_1] - T[g_2]\|_N \leq Kr \|g_1 - g_2\|_N = k \|g_1 - g_2\|_N$$

Where  $Kr = k < 1$  by choice of  $r$ . Hence T is a contraction, this implies that there exist one and only one function  $f(x) \in N$  such that

$f(x) = y_0 + \int_{x_0}^x F[s, f(s)]ds, f(x_0) = y_0$  is the unique solution of the differential equation (1) with the stated initial condition.

4.0 Conclusion: We have shown that we can apply the fixed point theorem to establish the existence of solution to the differential equation stated in (1). You are supposed to master the concept developed in this unit before proceeding to the next unit.

5.0. Summary: The contraction fixed point theorem applied in this unit enables us to develop a unique solution to the differential equation stated in (1). It is one of the most powerful theorems in mathematical analysis.

It can be extended to spaces of infinitely many dimensions. This is beyond the scope of this unit.

#### 6.0 Tutor Marked Assignments

Determine an interval  $(x_0 - r, x_0 + r)$  where the existence of solution to the following differential equations are guaranteed.

1.  $y' = y, y(0) = 1$
2.  $y' = y^3, y(0) = 2$
3.  $y' = xy + y^2, y(0) = 0$
4.  $y'_1 = y_1 + y_2, y_1(0) = -1, y_2(0) = 1$

#### 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.

#### UNIT3: The Method of Successive Approximations

1.0. Introduction

2.0. Objectives

3.0. Main Content

- 3.1. The Method of Successive Approximations
- 4.0. Conclusion
- 5.0. Summary
- 6.0 Tutor Marked Assignment
- 7.0. References/Further Readings

1.0 Introduction: The method of successive approximations is a refinement of the old device of trial and error. What has been added is control of the limiting process. We know how often the process must be repeated to bring the result with the desired limit of tolerance. The method of trial and error can be traced back to Sir Isaac Newton who was first to be concerned with approximate solution of algebraic equation. An infinite iteration process for the positive solution of the transcendental equation defined as:

$$x = \alpha \arctan x, \quad 1 < \alpha \dots\dots\dots \quad (\text{A})$$

was given by Joseph Fourier in his *Theorie Analytique de la Chaleur* of 1822. Fourier's argument is geometrical and highly intuitive. It is not difficult to give a strict analytic convergence proof.

The method of successive approximation was given by Emile Picard for differential equation in 1891. This method very soon became the standard method for proving existence and uniqueness theorems for all sorts of functional equations.

2.0 Objectives: At end of this unit you should be able to apply the method of successive approximation to determine existence and uniqueness of differential equation.

### 3.0 MAIN CONTENT

#### 3.1 The Method of Successive Approximations

Let us consider a vector differential equation defined by

$$y' = F(x, y), y(x_0) = y_0 \dots \dots \quad (1)$$

$F(x, y)$  is defined and continuous in:

$$B: |x - x_0| < a \quad \|y - y_0\| < b, \quad \|F(x, y)\| \leq M \quad (2)$$

$$\|F(x, y_1) - F(x, y_2)\| \leq K \|y_1 - y_2\| \quad (3)$$

We shall state the following theorem:

Theorem (1): There exists a unique function  $f(x)$ , on,  $R^1$  to,  $R^n$  defined for  $|x - x_0| < r$ , where

$$r < \min(a, \frac{b}{M}) \quad (4)$$

Proof: We replace the differential equation with the initial by the equivalent integral equation:

$$f(x) = y_0 + \int_{x_0}^x F[s, f(s)]ds \quad (5)$$

$$f_0(x) = y_0$$

Now define

$$f_m(x) = y_0 + \int_{x_0}^x F[s, f_{m-1}(s)]ds, m = 1, 2, 3, \dots \quad (6)$$

For these functions to be well defined, we restrict  $x$  to the interval  $(x_0 - r, x_0 + r)$ . Suppose it is known that for some value of  $m$ , the function  $f_{m-1}(x)$  is well defined in this interval. It is obvious that  $f_{m-1}(x) = y_0$ , but the induction hypothesis must also include that  $f_{m-1}(x)$  is continuous and  $\|f_{m-1}(s) - y_0\| < b$ . We then see that  $F[s, f_{m-1}(s)]$  is well defined and continuous. Further

$$\|F[s, f_{m-1}(s)]\| \leq M,$$

Hence

$\int_{x_0}^x F[s, f_{m-1}(s)]ds$ , exist as a continuous function of  $x$  and its norm does not exceed  $M|x - x_0| < Mr < b$  by the choice of  $r$ .

This implies that  $f_m(x)$  is also continuous and satisfies  $f_m(x_0) = y_0, \|f_m(x) - y_0\| < b$

It follows that the approximation are well defined for all  $m$ . To prove the existence of  $\lim f_m(x)$ , we resort to the Lipschitz condition. We have

$$\|f_m(x) - f_{m-1}(x)\| = \left\| \int_{x_0}^x \{F[s, f_{m-1}(s)] - F[s, f_{m-2}(s)]\} ds \right\| \leq K \left| \int_{x_0}^x \|f_{m-1}(s) - f_{m-2}(s)\| ds \right|$$

We know suppose that for some  $m$  we have the estimate

$$\|f_{m-1}(s) - f_{m-2}(s)\| \leq \frac{K^{m-2}}{(m-1)!} M |s - x_0|^{m-1}, |s - x_0| < r \quad (7)$$

This estimate is certainly very true for  $m = 2$ . we then get

$$\|f_m(x) - f_{m-1}(x)\| \leq \frac{K^{m-1}}{(m-1)!} M \left| \int_{x_0}^x |s - x_0|^{m-1} ds \right| = \frac{K^{m-1}}{m!} M |x - x_0|^m. \quad \text{Therefore the estimate is true for all } m$$

Hence the series

$$f_0(x) + \sum_{n=1}^{\infty} [f_n(x) - f_{n-1}(x)] \quad (8)$$

Whose partial sum is  $f_m(x)$ , converges in norm for  $|x - x_0| < r$  uniformly in  $x$ . Hence, it, sum,  $f(x)$ , is a continuous function on  $R^1 \longrightarrow R^n$ .

The strong uniform convergence of the vector series (8) obviously implies the absolute and uniform convergence of the  $n$  component series to continuous functions on  $R^1, to, R^n$  The estimate (7) obviously implies that

$$\|f(x) - f_m(x)\| \leq \frac{K^m}{m!} M \exp(K|x - x_0|) |x - x_0|^m \dots \dots \quad (9)$$

It is an easy matter to observe that if  $|x - x_0|$  is not large,  $f_m(x)$  converges rapidly to its limit  $f(x)$ . Therefore from the uniform convergence of  $f_m(x)$  to  $f(x)$  it follows that  $F[s, f_{m-1}(s)]$  converges to uniformly to  $F[s, f(s)]$  and

$$\int_{x_0}^x F[s, f_{m-1}(s)]ds \rightarrow \int_{x_0}^x F[s, f(s)]ds$$

uniformly in  $x$ . From (6) it follows that  $f(x)$  satisfies (5) and consequently also the differential equation and the initial condition. That this is the only solution also follows from the Lipschitz condition. So to prove uniqueness we may suppose that  $g(x)$  is a solution defined in some interval  $(x_0 - r_1, x_0 + r_1)$ . then

$$g(x) = y_0 + \int_{x_0}^x F[s, g(s)]ds, \text{ and if } |x - x_0| < \min(r, r_1) \text{ we have}$$

$$\|f(x) - g(x)\| = \left\| \int_{x_0}^x \{F[s, f(s)] - F[s, g(s)]\}ds \right\| \leq K \left| \int_{x_0}^x \|f(s) - g(s)\|ds \right|$$

Set  $h(x) = \|f(x) - g(x)\|$ , then  $h(x)$  is a continuous non-negative function that satisfies,

$0 \leq h(x) \leq K \left| \int_{x_0}^x h(s)ds \right|$  hence  $h(x)$  is identically 0. Therefore  $f(x)$  is the only solution of (1) with  $f(x_0) = y_0$

**4.0 Conclusion:** Various questions arise when we want to use theorem (1) above. The first of these concerns the effective determination of  $a, b$  and  $M$  and the verification of the Lipschitz condition. We leave this for future considerations. We have justified the existence of solution to functional differential equations; we have also proved the uniqueness of this solution. You are required to read carefully before proceeding to next unit.

**5.0 Summary:** We have proved the existence of functional differential equations by successive approximation methods. Successive approximation method is essentially an iterative method that needs to be carefully designed to give a solution to the differential equation under consideration. Once the equivalent integral equation of the given differential equation is known then it is just an easy matter to design the appropriately iterative scheme for the equation, which will eventually converge to the solution of the equation.

## 6.0 Tutor Marked Assignments

(1) Solve  $y' = y + x, y(0) = C$ , by method of successive approximation

(2) If  $y(x)$  is a solution of

$$y'' - x^2 y = 0, y(0) = y_{01}, y'(0) = y_{02},$$

Show that

$$y(x) = y_{01} + y_{02}x + \int_0^x (x-s)s^2 y(s) ds.$$

Use the method of successive approximations to find  $y(s)$  in the special case  $y_{01} = 1, y_{02} = 0$ , Take  $f(x) \equiv 1$

(3) The Thomas-Fermi equation defined by

$$x^{1/2} y'' = y^{3/2}$$

arises in nuclear physics. Show that it has a solution of the form  $Cx^\alpha$ . Show also that it can be transformed into a system to which method of successive approximation can be applied so that there solution in some interval  $[0,r]$  satisfying an initial condition of the form  $y(0) = a > 0, y'(0) = b$

## 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London

## MODULE 2: SPECIAL FUNCTIONS

### UNIT1: Special Functions

#### 1.0. Introduction

#### 2.0. Objectives

#### 3.0. Main Content

#### 3.1. Special Functions

##### 3.1.1 Gamma function

##### 3.1.2 Beta function

##### 3.1.3 Factorial Notation

#### 4.0. Conclusion

#### 5.0. Summary

#### 6.0 Tutor Marked Assignment

#### 7.0. References/Further Readings

1.0 Introduction: In this unit we shall examine some special functions such as Beta function, Gamma function and Factorial function. These functions are of very useful mathematical importance in solving differential equations and other applied mathematics problems.

2.0 At the end of this unit you should be able to :

- define beta function, gamma function, and factorial notations
- apply these functions to solve mathematical problems

### 3.0 MAIN CONTENT

#### 3.1. Special Functions

##### 3.1.1 Gamma functions

One of the most important functions is the gamma function, written and defined by the integral

$$(1) \quad \Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$$

(More generally, if we consider also complex values, for those  $\alpha$  whose real part is positive).

By integration by parts, we find

$$\Gamma(\alpha + 1) = \int_0^{\infty} e^{-t} t^{\alpha} dt = -e^{-t} t^{\alpha} \Big|_0^{\infty} + \alpha \int_0^{\infty} e^{-t} t^{\alpha-1} dt = \alpha \Gamma(\alpha)$$

Thus we obtain the important functional relation of the gamma function

$$(2) \quad \Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$$

Let us suppose that  $\alpha$  is a positive integer, say,  $n$ . Then repeated application of (2) yields

$$\begin{aligned} \Gamma(n + 1) &= n \Gamma(n) \\ &= n(n-1) \Gamma(n-1) \\ &\dots\dots\dots \\ &= n(n-1)\dots\dots\dots 3 \cdot 2 \cdot 1 \end{aligned}$$

Now  $\Gamma(1) = \int_0^{\infty} e^{-t} dt = -e^{-t} \Big|_0^{\infty} = 1$

$$(3) \quad \therefore \Gamma(n+1) = n!$$

$\therefore$

Hence gamma function can be regarded as a generalization of the factorial function

By repeated application of (2)

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{\alpha} = \frac{\Gamma(\alpha + 2)}{\alpha(\alpha + 1)} = \dots\dots\dots \frac{\Gamma(\alpha + k + 1)}{\alpha(\alpha + 1)\dots(\alpha + k)}$$

Thus we obtain the relation

$$(5) \quad \Gamma(\alpha) = \lim_{k \rightarrow \infty} \frac{\Gamma(\alpha + k + 1)}{(\alpha)(\alpha + 1)\dots(\alpha + k)} \quad (\alpha \neq 0, -1, -2, \dots)$$

Gauss defined Gamma function as follows

$$(6) \quad \Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n! n^{\alpha-1}}{\alpha(\alpha + 1)\dots(\alpha + n)}$$

Where

**Problem 1.**  $\alpha > 0$  and  $n$  is a *ve* integer, then

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \int_0^n \left\{ 1 - \frac{t}{n} \right\}^n t^{\alpha-1} dt.$$

**Proof:** Now consider the integral

$$\int_0^n \left\{ 1 - \frac{t}{n} \right\}^n t^{\alpha-1} dt.$$

Substitute  $t = nx$  in the integral, we obtain

$$\int_0^n \left\{ 1 - \frac{t}{n} \right\}^n t^{\alpha-1} dt = n^\alpha \int_0^1 (1-x)^n x^{\alpha-1} dx$$

By integrating by parts gives the formula

$$\int_0^n (1-x)x^{\alpha-1} dx = \frac{n}{\alpha} \int_0^1 (1-x)^{n-1} x^\alpha dx$$

Repeating integration by parts, we get

$$\int_0^n (1-x)^n x^{\alpha-1} dx = \frac{n(n-1)(n-2)\dots 1}{\alpha(\alpha+1)\dots(\alpha+n-1)} \int_0^1 x^{\alpha+n-1} dx$$

Thus

$$\int_0^n \left( 1 - \frac{t}{n} \right)^n t^{\alpha-1} dt = \frac{n! n^\alpha}{\alpha(\alpha+1)\dots(\alpha+n-1)}$$

$$\therefore dt \int_0^n \left\{ \left\{ 1 - \frac{t}{n} \right\}^n t^{\alpha-1} dt \right\} = \lim_{n \rightarrow \infty} \frac{n! n^\alpha}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)} = \Gamma(\alpha)$$

**Lemma 1.** If  $0 \leq \alpha < 1$ ,  $1 + \alpha \leq \exp \alpha \leq (1 - \alpha)^{-1}$ , compare the three series.

$$(1 + \alpha)^{-1} = 1 + \alpha, \quad \exp(\alpha) = 1 + \alpha + \sum_{N=2}^{\infty} \frac{\alpha^N}{N!}$$

$$(1 + \alpha)^{-1} = 1 + \alpha + \sum_{N=2}^{\infty} a^N$$

**Lemma 2.** If  $0 \leq \alpha < 1$ ,  $(1 - \alpha)^n \geq 1 - n\alpha$ , for a position integer

**Proof:** For  $n = 1$ ,  $1 - \alpha = 1 - \alpha$ , as derived.

Assume that

$$(1 - \alpha)\beta \geq 1 - \beta\alpha,$$

Multiply each member by  $1 - \alpha$ , to obtain

$$(1 - \alpha)^{\beta+1} \geq (1 - \alpha)(1 - \beta\alpha) = 1 - (b + 1)\alpha + b\alpha^2$$

So that

$$(1 - \alpha)^{\beta+1} \geq 1 - (b + 1)\alpha = 1 - (b + 1)\alpha + b\alpha^2$$

Lemma 2, follows by induction

Lemma 3. If  $0 \leq t < n$ ,  $n$  a positive integer

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n!}$$

Proof: In Lemma I, put  $\alpha = \frac{t}{n}$ , we get

$$\left(1 + \frac{t}{n}\right) \leq e^{\frac{t}{n}} \leq \left(1 - \frac{t}{n}\right)^{-1}$$

From which

$$(a) \left(1 + \frac{t}{n}\right)^n \leq e^{-t} \frac{t}{n} \leq \left(1 - \frac{t}{n}\right)^{-n}$$

Or

$$\left(1 + \frac{t}{n}\right)^n \geq e^{-t} \frac{t}{n} \geq \left(1 - \frac{t}{n}\right)^{-n}$$

So that

$$e^t - \left(1 - \frac{t}{n}\right)^n \geq 0$$

$$\therefore e^t - \left(1 - \frac{t}{n}\right)^n \leq e^t \left[1 - \left(1 - \frac{t^2}{n^2}\right)\right]$$

But by (a)

$$e^t \geq \left(1 - \frac{t}{n}\right)^n$$

$$\therefore e^t - \left(1 - \frac{t}{n}\right)^n \leq e^{-t} \left[1 - \frac{t^2}{n^2}\right]^2$$

In lemma 2, we have shown that

$$(1 - \alpha)^n \geq 1 - n\alpha.$$

$$\therefore \left(1 - \frac{t^2}{n^2}\right)^n \geq 1 - n\alpha$$

$$\therefore e^t - \left(1 - \frac{t}{n}\right)^n \leq \frac{e^{-t} t^2}{n}$$

Problem 2. Show that the two definitions of gamma function are equivalent.

Proof: By using Gauss's definition, we proved that

$$\Gamma(z) = \lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$$

Now

$$= \lim_{n \rightarrow \infty} \left[ \int_0^n \left[ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dx \right]$$

From the convergence of the integral

$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$$

It follows

$$= \lim_{n \rightarrow \infty} \int_0^n e^{-t} - t^{z-1} dt = 0$$

Hence

$$\int_0^\infty e^{-t} t^{z-1} dt - \Gamma(z) + \lim_{n \rightarrow \infty} \int_0^n \left[ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt$$

Now

$$\int_0^\infty e^{-t} t^{z-1} dt \text{ Converges, so } \int_0^n e^{-t} t^{z-1} dx$$

is bounded.

Thus

$$\lim_{n \rightarrow \infty} \int_0^n \left[ e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt = 0$$

$$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$$

Problem 3. Show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (z \neq 0, \neq 1, \neq 2, \dots)$$

Proof: using Gauss definition of gamma function

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!}{z(z+1)(z+2)\dots(z+n)}$$

$$z \neq \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{s^2}\right)$$

$$= \frac{\sin \pi z}{\pi}$$

Note if we put  $z = 1/2$ , we get

$$\frac{1}{[\Gamma(1/2)]^2} = \frac{1}{\pi}$$

or

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

**Problem 3. Show that**

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (2z = 0, -1, -2, \dots)$$

**Proof :**

$$\frac{2^{2z} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)}{\Gamma(2z)}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{2^{2z} n^{z-1} n^{z-1} + \frac{1}{2}}{z(z+1)\dots(z+n)\left(z + \frac{1}{2}\right)\left(2 + \frac{3}{2}\right)\dots\left(z + n + \frac{1}{2}\right)} \right\}$$

$$= \lim_{n \rightarrow \infty} \left[ \frac{(n!)^2 2^{2n+1}}{(2n)! \sqrt{n}} \right]$$

The last quantity is independent of  $z$  and must be finite since the left side exists.

$$\therefore \frac{2^{2z} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)}{\Gamma(2z)} = A$$

$\therefore$

Put  $z = 1/2$

We have

$$A = 2\sqrt{\pi}$$

$$\therefore \Gamma(2z) = \frac{2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)}{\sqrt{\pi}} \therefore$$

### 3.1.2 Beta -function:

We define in Beta-function  $B(p, q)$  by

$$(1) B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dx, R(p) > 0, R(q) > 0$$

Another useful form of this function can be obtained by putting  $t = \sin^2 \theta$ , thus arriving at

$$(2) B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1} \theta \cos^{2q-1} \theta d\theta, R(p) > 0, R(q) > 0$$

Next we establish the relation between gamma and beta-functions

**Problem:** If  $R(p) > 0, R(q) > 0$ . Then

$$B(p, q) = \frac{r(p)r(q)}{r(p+q)}$$

Proof:  $r(p)r(q) = \int_0^\infty e^{-t} t^{p-1} dt \int_0^\infty e^{-u} u^{q-1} du$

Substituting  $t = x^2$  and  $U = y^2$  it gives  $r(p)r(q) = 4 \int_0^\infty e^{-x^2} x^{2p-1} dx \int_0^\infty e^{-y^2} y^{2q-1} dy$

$$r(p)r(q) = 4 \int_0^\infty \int_0^\infty \exp(-x^2 - y^2) x^{2p-1} y^{2q-1} dx dy$$

Next turn to polar co-ordinate for the iterated integration over the first quadrant in xy-plan.  $r(p)r(q) = 4 \int_0^{\frac{\pi}{2}} \int_0^\infty \exp(-r^2) r^{2p+2q-2} \cos^{2p-1} \theta \sin^{2q-1} \theta r dr d\theta$

$$2 \int_0^\infty \exp(-r^2) r^{2p+2q-1} dr \int_0^{\frac{\pi}{2}} \cos^{2p-1} \theta \sin^{2q-1} \theta d\theta$$

Take  $r^{2t}$  and  $\theta = \frac{1}{2}\pi - \theta$ , we obtain

$$r(p)r(q) = \int_0^\infty \exp(-t) t^{p+q-1} dt \int_0^{\frac{1}{2}\pi} \sin^{2p-1} \theta \cos^{2q-1} \theta d\theta$$

$$= r(p+q)B(p, q)$$

$$\therefore B(p, q) = \frac{r(p)r(q)}{r(p+q)}$$

### 3.1.3 Factorial notations

$$\begin{aligned} (1) \quad (\alpha)_n &= \frac{n!}{l!} (\alpha + k - 1) \\ &= \alpha(\alpha+1)\dots(\alpha+n-1) \\ (\alpha)_0 &= 1, \alpha \neq 0 \end{aligned}$$

The function  $(\alpha)_0$  is called the factorial notation

Problem: Show that

$$(\alpha)_{2n} = 2^{2n} \left( \frac{\alpha}{2} \right)_n \left( \frac{\alpha+1}{2} \right)_n$$

Proof:-

$$\begin{aligned} (\alpha)_{2n} &= (\alpha)(\alpha+1)(\alpha+2)(\alpha+3)\dots(\alpha+2n-1) \\ &= [ (\alpha)(\alpha+2)\dots(\alpha+2n-2) ] [ (\alpha+1)(\alpha+3)\dots(\alpha+2n-1) ] \\ &= 2^{2n} \left[ \left( \frac{\alpha}{2} \right) \left( \frac{\alpha+2}{2} \right) \left( \frac{\alpha+4}{2} \right) \dots \left( \frac{\alpha+2n-2}{2} \right) \right] \\ &\quad \left[ \left( \frac{\alpha+1}{2} \right) \left( \frac{\alpha+1}{2} + 1 \right) \dots \left( \frac{\alpha+1}{2} + n - 1 \right) \right] \\ &= 2^{2n} \left( \frac{\alpha}{2} \right)_n \left( \frac{\alpha+1}{2} \right)_n \end{aligned}$$

Similarly we can show that

$$(\alpha)_{kn} = K^{kn} \left( \frac{\alpha}{k} \right)_n \left( \frac{\alpha+1}{k} \right)_n \dots \left( \frac{\alpha+k-1}{k} \right)_n$$

Problem show that

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

Proof:

$$\begin{aligned} \Gamma(\alpha+n) &= (\alpha+n-1)(\alpha+n-2)\dots\alpha\Gamma(\alpha) \\ &= (\alpha)(\alpha+1)\dots(\alpha+n-1)\Gamma(\alpha) \\ \Gamma(\alpha+n) &= (\alpha)_n \Gamma(\alpha) \\ \therefore (\alpha)_n &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \end{aligned}$$

**4.0 Conclusion:** In this unit we have studied Gamma function, Beta function and Factorial notations. You are required to study these functions because of immediate applications in future.

**5.0 Summary:** The study of special functions in mathematics is of significant importance in mathematics. Study this area properly before moving to the next unit.

### 6.0 Tutor Marked Assignment

1. The Beta function of  $p, \text{ and } q$  is defined by the integral

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt, (p, q > 0).$$

By writing  $t = \sin^2 \theta$  obtain the equivalent form

$$B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1} \theta \cos^{2q-1} \theta d\theta, (p, q > 0)$$

2. Show that

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

3. By writing  $t = x/(x+a)$  in the definition of Beta function, show that

$$\int_0^{\infty} \frac{x^{p-1} dx}{(x+a)^{p+q}} = a^{-q} B(p, q)$$

#### 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London

## UNIT2: Hyper geometric Function

- 1.0. Introduction
- 2.0. Objectives
- 3.0. Main Content
- 3.1. Hypergeometric Functions
- 4.0. Conclusion
- 5.0. Summary
- 6.0 Tutor Marked Assignment
- 7.0. References/Further Readings

1.0. Introduction: In this unit we shall consider a class of function usually refers to as hyper-geometric functions. The series solution of the associated differential equation usually takes the form of a geometric series. Most often hyper-geometric equation has  $x = 0, x = 1$  and  $x = \infty$  as regular points and ordinary point elsewhere.

2.0. Objectives: At the end of this unit you should have learnt about the differential equations that can give rise to hyper-geometric functions.

- learnt about the properties of this functions
- be able to apply this function where necessary.

### 3.0. MAIN CONTENT.

#### 3.1 Hyper-geometric Function:

The solutions of the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby = 0 \quad (1)$$

are generally called Hyper geometric functions.

Note that  $a, b,$  and  $c$  are fixed parameters.

We solve this equation (1) about the regular singular point  $x=0$

Shifting the index

$$\sum_{n=0}^{\infty} n(n+c-1)x^{n-1} - \sum_{n=1}^{\infty} (v+a-1)(n+b-1)e_{n-1}x^{n-1} = 0$$

For  $n \geq 1$

$$e_n = \left( \frac{a(a+a)(n+b-1)}{n(n+c-1)} \right) e_{n-1}$$

$$e_n = \left( \frac{a(a+a)(a+2)\dots(a+n-1).b(b+1)(b+2)\dots(b+n-1)e_0}{n! . c(c+1)(c+2)(c+2)\dots(c+n-1)} \right)$$

Using factorial notation, we have

$$e_n = \frac{(a)_n (b)_n}{n! (c)_n} e_0$$

Let us choose  $e_0 = 1$

$$\begin{aligned} y_1 &= 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} x^n \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \end{aligned}$$

We have the symbol

${}_2F_1(a, b, c, x)$  to represent solution

$$y_1 = \sum_{n=0}^{\infty} \frac{(a)(b)_n}{(c)_n n!} x^n$$

$\therefore$

$${}_2F_1(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n.$$

The solution is valid in  $0 < |x| < 1$ . The other root of the indicial equation is (i-

c). We may put  $y = \sum_{n=1}^{\infty} f_n x^{n+1-c}$

For the moment let  $c$  be not an integer for (1), the indicial equation has root

zero and  $i-c$ . Let  $y = \sum_{n=1}^{\infty} e_n x^{n+r}$

$$\begin{aligned} \sum_{n=1}^{\infty} e_n x^{n+b} (n+b-1)x^{n+b} - \sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c \sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c \sum_{n=0}^{\infty} e_n (n+b)x^{n+b-1} \\ - (a+b+1) \sum_{n=0}^{\infty} e_n x^{n+b=0} \end{aligned}$$

or

$$\sum_{n=0}^{\infty} e_n (n+b)(n+b-1+c)x^{n+b-1}$$

$$\sum_{n=0}^{\infty} e_n [ ab+1)(a+b+1)(n+b)(n+b)(n+b-1) ] x^{n+b} = 0$$

The indicial equation is

$$e_n (b)(b-1+c) = 0$$

(Note c is not a integer).

Corresponding to  $b = 0$ ,

$$\sum_{n=1}^{\infty} n(n+c-1)e_n = \sum_{n=0}^{\infty} (n+a)(n+b)e_n x^n = 0$$

**Problem 1:** If  $R(c-a-b) > 0$  and if c is neither zero nor a negative integer,

$${}_2F_1(a, b, c, 1) = \frac{r(c)r(c-a-b)}{r(c-a)r(c-b)}$$

**Proof**

$$\begin{aligned} & {}_2F_1(a, b, c, 1) \\ &= \frac{r(c)}{r(b)r(c-b)} \\ &= \frac{r(c)}{r(b)r(c-b)} \frac{r(b)r(c-a-b)}{r(c-a)} \end{aligned}$$

**Problem 2:** Show that

$$(a) \quad {}_2F(\alpha, \beta, \beta, x) = (1-x)^{-\alpha}$$

$$(b) \quad x {}_2F(1; 1; 2; -x) = \text{Log}(1+x)$$

**Solution:**

$$(a) \quad {}_2F(\alpha, \beta, \beta, x)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(x)_n}{n!} x^n \\ &= 1 + \alpha x + \frac{(\alpha+1)}{2!} x^2 + \dots + \frac{\alpha(\alpha+1)(\alpha+1)}{3!} x^3 + \dots = (1-x)^{-\alpha} \end{aligned}$$

$$(b) \quad x {}_2F(1; 1; 2; -x) = \text{Log}(1+x)$$

$$x \left[ 1 + \frac{1.1}{1.2} (-x) + \frac{1.2.1.2}{1.2.2.3} (-x)^2 + \frac{1.2.3.1.2.3}{1.2.3.2.34} (-x)^3 \right]$$

$$y_2 = x^{1-c} {}_2F_1(a+1-c, b+1-c; 2-c; x)$$

**Problem:** If  $1 \leq 1 \leq 1$ , and if  $R(c) > R(b) > 0$ ,

$${}_2F_1(a; b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt$$

**Proof**

**Beta-function Now**

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \int_0^1 t^{b-1} (1-t)^{c-b-1} dt$$

Also

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$

Thus

$$\begin{aligned} {}_2F_1(a; b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \int_0^1 t^{b+n-1} (1-t)^{c-b-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} \sum_{n=0}^{\infty} \frac{(a)(zt)^n}{n!} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-zt)^a dt \end{aligned}$$

Where

$$\begin{aligned} (1-z)^a &= \sum_{n=0}^{\infty} \frac{(-\alpha)(-\alpha-1)\dots(-\alpha-n+1)(-1)^n y^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha+1)\dots(\alpha+n-1)y^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{a(\alpha+1)\dots(\alpha+n-1)y^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n y^n}{n!} \end{aligned}$$

4.0 Conclusion: You have learnt in this unit some properties of hyper geometric functions. You are requested to study this unit properly before going to the next unit.

5.0 Summary: Recall that you learnt about the class of differential equation, that usually give rise to hyper-geometric functions. You also learn about the relations of this function to Gamma and Beta functions. Study this unit properly before going to the next unit.

6.0 Tutor Marked Assignment:

(1) If  $\text{Re}(c - a - b) > 0$  and if  $c$  is neither zero nor a negative integer show that

$${}_2F_1(a; b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

(2) Show that

(a)  ${}_2F_1(\partial; \beta; \beta; x) = (1-x)^{-\alpha}$

(b)  $x^2 {}_2F_1(1; 1; 2; -x) = \log(1+x)$

## 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London

## UNIT3: BESSEL FUNCTIONS

1.0. Introduction

2.0. Objectives

3.0. Main Content

3.1. Bessel- Functions

3.1.1 Bessel functions of the first kind.

4.0. Conclusion

5.0. Summary

6.0 Tutor Marked Assignment

7.0. References/Further Readings

1.0. Introduction: In solving differential equation we often come across some problems which exhibit some characteristic which needed to be studied further. Such equations are, Legendry equation, Bessel equations e.t.c. We shall study in detail in this unit, the Bessel equation which give rise to Bessel functions. This is because of the wide applicability of this functions, in physics and applied mathematics.

2.0. Objectives: At the end of this unit you be able to

- identify Bessel functions correctly
- solve problems related to Bessel functions

### 3.0 MAIN CONTENT

#### 3.1: Bessel function

The equation

$$(1) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2)y = 0$$

is called Bessel's equation of index  $v$ .

- (i)  $x = 0$  is the regular Singular point of the equation (1) in the finite plane
- (ii) Assume that  $v$  is not integer.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting this expression and its first and second derivatives into Bessel equation, we obtain

$$= \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - v^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$(a) \quad r(r-1)c_0 + rc_0 - v^2 c_0 = 0 \quad (m=0)$$

$$(b) \quad (r+1)(r)c_1 + (r+1)c_1 - v^2 c_1 = 0 \quad (m=1)$$

$$(c) \quad (m+r)(m+r-1)c_m + c_{m-2} - vc_m = 0 \quad (m=2,3,\dots)$$

Now  $c_0 \neq 0$ , thus the indicial equation from (a)  $(r+v)(r-v) = 0$

The roots are  $r = v - v = 0$

$$2_1 - r_2 = 2v$$

$$v \neq 0,$$

$$v \neq \text{integer}$$

$$2v \text{ integral multiply of } 2v, \text{ i.e. } v \text{ is zero or } +ve \text{ integer}$$

Now we obtain the solution corresponding to the value  $r = v$ .

Form (b) we obtain  $c_1 = 0$  (c) can be written

$$(m+r-v)(m+r+v)c_m + c_{m-2} = 0$$

Since  $c_1 = 0$ , it follows that  $c_3 = c_5 = c_7 = \dots = 0$ . Thus we put replace  $m$  by  $2m$ .

$$(2m+r-v)(2m+r+v)c_m + c_{2m-2} = 0$$

Now  $r = v$

$$(2m + 2v)(2m)c_{2m} + c_{2m-2} = 0$$

$$\therefore c_{2m} = -\frac{c_{2m-2}}{2^2(v+m)m} \quad (\text{but } v \text{ is not integer}) \quad (m = 1, 2, \dots)$$

Assume

$$c_0 = \frac{1}{v_2 r(v+1)}$$

$$c_2 = \frac{c_0}{2^2(v+1)} = \frac{1}{2^{v+2} 1! r(v+2)}$$

$$c_4 = \frac{c_2}{2 \cdot 2^2(v+2)} = \frac{1}{2^{v+2} 2! r(v+3)}$$

$$c_{2m} = -\frac{(-1)^m}{2^{2m+v} m! r(v+m+1)}$$

Thus the solution is

$$y = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m! r(v+m)}$$

We denote this solution by the notation

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} r(m+\nu+1)m!} \quad (2)$$

$J_\nu(x)$  is called the Bessel Function of the first kind of order  $\nu$ .

By Ratio test we know that the series converges for all values of  $x$

Replacing  $\nu$  by  $-\nu$ , we have

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} r(m-\nu+1)m!} \quad (3)$$

(2) and (3) are the independent solutions.

Thus  $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

(i) If  $\nu = 0$ , then the solution  $J_\nu(x)$  and  $J_{-\nu}(x)$  are identical. One can verify from (2) and (3)

(ii) If  $\nu$  is  $+\nu e$  integer, then the second solution  $J_{-\nu}(x)$  is not independent of  $J_\nu(x)$

Say  $\nu = n$  then the factor

$$\frac{1}{r(m-n+1)} = \frac{1}{m-n!} \text{ in (3) is zero}$$

When  $m < n$  Hence (3) is equivalent to

Replace  $m$  by  $m+n$  in 5, we get change the index

$$J_{-v}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} \left(\frac{x}{2}\right)^{2m+n}}{m!m+n!} \quad (6)$$

From (2), when  $v = n$  integer, thus

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+n}}{m!, m+n!} \quad (7)$$

From (6) and (7), we get

$$J_{-v}(x) = (-1)^n J_n(x) \quad (8)$$

Further properties of Bessel functions of first kind

From (2)

$$x^v J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v}}{2^{2m+v} r(m+v+1)m!}$$

Now we use the formula

$$\alpha r(\alpha) = r(\alpha+1)$$

$$\begin{aligned} x^{v-1} x^v &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v-1}}{2^{2m+v} r(m+v)m!} \\ &= x^{v-1} x^v = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v-1} m! r(m+v)} \\ &= x^v J_{v-1}(x) \end{aligned}$$

Thus we obtain

$$\frac{d}{dx} [x^v J_v(x)] = x^v J_{v-1}(x) \quad (9)$$

Similarly, we can show that

$$\frac{d}{dx} [x^{-v} J_v(x)] = (-1)x^{-v} J_{v+1}(x) \quad (10)$$

(9) can also use written

$$vx^{v-1} J_v(x) + x^v J_v'(x) = x^v J_{v-1}(x) \quad (11)$$

(10) Can also be written

$$-vx^{-v-1} J_v(x) + x^{-v} J_v'(x) = -x^{-v} J_{v+1}(x) \quad (12)$$

Multiplying (12) by  $x^{2v}$  and subtracting from (11), we have

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x) \quad (13)$$

Multiplying (12) by  $x^{2v}$  and adding with (11), we get

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x) \quad (14)$$

(i) we know that

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{k+1}} \end{aligned}$$

Now  $2k+1 = r(2k+2) = (2)_{2k}$

$$2^{2k} + (1) = (2)_{2k}$$

$$2^{2k} + k! \left( \frac{3}{2} \right)_k$$

$$2^{2k} + k! \left( \frac{3}{2} \right)_k$$

$$\frac{2^{2k} k! r \left( k+1 + \frac{1}{2} \right)}{r \left( \frac{3}{2} \right)}$$

But  $r \left( \frac{3}{2} \right) = \frac{1}{2} r \left( \frac{1}{2} \right) = \frac{1}{2} \sqrt{\pi}$

$$\therefore 2k+1 = \frac{2^{2k+1} k! r \left( k+1 + \frac{1}{2} \right)}{\sqrt{\pi}}$$

$$\therefore \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1} \sqrt{\pi}}{2^{2k+1} k! r \left( k+1 + \frac{1}{2} \right)} \quad (15)$$

If we take  $\nu = \frac{1}{2}$ , then from (2), we have

$$J_{\nu}(x) = x \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k + \frac{1}{2}} r \left( k + \frac{1}{2} \right)} \quad (16)$$

From (15) and (16), we have

$$J_{\nu}(x) = \left( \frac{2}{\pi x} \right) \sin x$$

In the similar manner, by considering the expansion

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \text{ we obtain}$$

The formula

$$J_{-\nu/2}(x) = \left( \frac{2}{\pi x} \right) \cos x$$

Problems

(i)  $J_0(x) = -1J_1(x)$

(ii)  $J_n'' = \frac{1}{4}(J_{n-2} - 2J_n + J_{n+2})$

(iii)  $J_1(x) = J_0(x) - \frac{1}{x}J_1(x)$

(iv)  $J_2(x) = (1 - \frac{4}{x^2})J_1(x) - \frac{2}{x}J_0(x)$

(v)  $\int x^m J_n(x) dx = x^m J_{n+1}(x) - (m-n-1) \int x^{m-1} J_{n+1}(x) dx$

Solution

$$\int x^m J_n(x) dx = x^{m-n-1} [x^{n+1} J_n(x)] dx$$

$$\int x^{m-n-1} \frac{d}{dx} [x^{n+1} J_{n+1}(x)] dx$$

Integrating by parts, we have

$$= x^m J_{n+1}(x) - (m-n-1) \int x^{m-1} J_{n+1}(x) dx$$

This proves the result

Prove that

(vi)  $\int J_n(x) dx = -x^m - J_{n-1} + (m+n-1) \int x^{m-1} J_{n-1}(x) dx$

(vii)  $\int J_{v+1}(x) dx = \int J_{v-1}(x) dx - 2J_v(x)$

It immediately follows from the identity  $2J'_v(x) = J_{v-1}(x) - J_{v+1}(x)$

(viii)  $\int J_{v-1}(x) dx = -x^{1-v} J_{v-1}(x) + c$

(ix)  $\int x^{1-v} J_{v1}(x) dx = -x^{1-v} J_{v-1}(x) + c$

(x)  $\int x^3 J_0(x) dx = -x^3 J_1(x) - 2^2 x J_2(x) + c$

Problem: Defining the Bessel function  $J_n(x)$  by means of the general function

Example  $\exp\{\frac{1}{2}x(t-t^{-1})\} = \sum_{n=-\infty}^{\infty} J_n(x)$  show that,

If  $n$  is an integer

(a)  $J_n(x) = (\frac{1}{2}x)^n \sum_{r=0}^{\infty} \frac{(-x \frac{x^2}{4})^r}{r!(n+r)!}$

(b)  $J_{-n}(x) = (-1)^n J_n(x)$

(c)  $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

(d)  $J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)$

Solution

(a) Replace  $t$  by  $-\frac{1}{t}$  in the definition

$$\begin{aligned} \exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} &= \sum_{n=-\infty}^{\infty} (-1)^n J_n(x) \\ &= \sum_{n=-\infty}^{\infty} (-1)^{-n} t^n J_{-n}(x) \\ &= \sum_{n=-\infty}^{\infty} t^n J_n(x) \end{aligned}$$

Thus we get

$$J_n(x) + (-1)^n J_{n+1} = J_{-n}(x)$$

(a) The exponential on the left can be expressed as a product of two exponentials

$$\begin{aligned} \exp\left[\frac{x}{2}(t - t^{-1})\right] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{2}\right)^n t^{-n} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{x}{2}\right)^m t^{m-m} \end{aligned}$$

Every product of a term of the first series by a term of the second contains a factor  $t^{n-m}$ . Let us associate with each term of the second series the term of the first series corresponding to  $n = p + m$  ( $p > 0$ ). The product contains the factors  $t^p$ . Therefore the series expansions of the coefficient  $j_p(x)$  with +ve  $p$  follows.

By associating with each term of the first series the term of the second series which corresponds to  $m = p + n$  ( $p > 0$ ), the product contains now the factor  $t^{-p}$ . Thus  $j_p(x)$  is obtained.

Problem: Prove that

$$j_0(z) = \frac{1}{2} \int_0^{2\pi} \cos(z \cos \theta) d\theta$$

Proof : We know that

$$\exp\left[\frac{z}{2}(t - t^{-1})\right] = \sum_{n=-\infty}^{\infty} j_n(z)t^n$$

Put

$t = ie^{-i\theta}$ , take real parts of both sides and integrate between 0 and  $2\pi$  .....

$$\exp\left[\frac{z}{2}(ie^{-i\theta} + ie^{+i\theta})\right] = \sum_{n=-\infty}^{\infty} j_n(z)i^n(\cos\theta + i\sin\theta)$$

$$j_0(z) + \sum_{n=1}^{\infty} j_n(z)i^n(\cos\theta + i\sin\theta)$$

$$+ \sum_{n=1}^{\infty} j_n(z)i^n(\cos\theta + i\sin\theta)$$

$$\exp[z\cos\theta] + J_0(z) + \dots$$

Problem prove that

$$\int_0^{\pi/2} J_0(z\cos\theta)\cos\theta d\theta = \frac{\sin z}{z}$$

Solution

$$j_p(z) = \left(\frac{z}{2}\right)^p \sum_{m=1}^{\infty} \frac{(-1)^m}{m!m+p!} \left(\frac{z}{2}\right)^{2m}$$

Put  $p = 0$

$$j_0(z) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!m!} \left(\frac{z}{2}\right)^{2m}$$

Replace  $2$  by  $z\cos\theta$  multiply both sides of integrate between 0 to

$$\int_0^{\pi/2} j_0(z\cos\theta)\cos\theta d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{m! m! 2^{2m}} \int_0^{\frac{\pi}{2}} j_0(\cos \theta)^{2m} d\theta$$

Now  $\int_0^{\frac{\pi}{2}} (\cos \theta) d\theta = \frac{2^{2m} (m!)^2}{(2m+1)!}$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m z^{2m}}{(2m+1)!}$$

Now we know that

$$\begin{aligned} \sin \theta &= \theta - \frac{\theta^3}{13} + \frac{\theta^5}{15} \dots\dots\dots \\ \frac{\sin \theta}{\theta} &= 1 - \frac{\theta^2}{13} + \frac{\theta^4}{15} \dots\dots\dots \\ &= \frac{\sin \theta}{z} \end{aligned}$$

### 3.1.1 Bessel Functions of the first Kind

In the definition of Bessel function  $j_p(z)$  put  $z = iy$ , then ( $p$  integer)

$$\begin{aligned} j_p(iy) &= e^{+i\frac{p\pi}{2}} I_p(y) \\ j_p(iy) &= e^{+i\frac{p\pi}{2}} I_p\left(\frac{iy}{2}\right)^p \sum_{m=0}^{\infty} \frac{1}{m! m! + p_1} \left(\frac{y}{2}\right)^{2m} \end{aligned}$$

(2) Bessel function of the second kind

Solution

$$\int_0^2 t J_n(at) J_n(bt) dt$$

$$\frac{z \{ a J_n(bz) J_n'(az) - b J_n(az) J_n'(bz) \}}{b^2 - a^2}$$

Solution

$$2^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - n^2) y = 0$$

$$y_1 = J_n(at), y_2 = J_n(bt)$$

(i)  $t^2 y'' + ty_1' + (a^2 t^2 - n^2) y_1 = 0$   
 $t^2 y_2'' + ty_2' + (b^2 t^2 - n^2) y_2 = 0$

by  $y_2$  and (2) by  $y_1$ , and subtracts, we find

find

$$(y_2 y_2'') + t(y_2 y_1 - y_1 y_2'(b^2 - a^2)t^2 y_1 y_2$$

or

$$\frac{d}{dt} \{t(y_2 y_1'' - y_1 y_2') + t(y_2 y_1 - y_1 y_2')\} = (b^2 - a^2) t y_1 y_2'$$

or

$$\frac{d}{dt} \{t(y_2 y_1'' - y_1 y_2')\} = (b^2 - a^2) t y_1 y_2'$$

Integrating with subtract to  $t$  from 0 to  $z$  yield

$$(b^2 - a^2) \int_0^z t(y_2 y_1 dt = t(y_2 y_1 - y_1 y_2')$$

4.0 Conclusion: We have considered Bessel function in its general setting in this unit . You are required to read this unit carefully before going to the next unit.

5.0 Summary: Recall that Bessel functions are usually associated with a class of equations called Bessel equations. They are usually denoted by the notation:

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} r(m+\nu+1)m!}$$

We gave some examples in this unit to enable you understand the content of this unit .In this unit we have also examined another type of Bessel function usually refer to as Bessel Function of the first Kind. However you are to master this unit properly before attempting to read the next unit.

6.0 Tutor Marked Assignment:

1. Given that

$$e^{\frac{x}{2}(r-\frac{1}{r})} = \sum_{n=-\infty}^{\infty} r^n J_n(x)$$

Deduce that  $(n+1)J_{n+1}(x) = \frac{x}{2}[J_n(x) + J_{n+2}(x)]$

2. Obtain the general solution of each of the following equations in terms of Bessel functions, or if possible in terms of elementary functions

$$(a) x \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + xy = 0 \quad (b) x \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 4x^3 y = 0 \quad (c) x^4 \frac{d^2 y}{dx^2} + a^2 y = 0$$

## 7.0 REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

## MODULE 3: SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATION

### UNIT1: Legendry Function

#### 1.0. Introduction

#### 2.0. Objectives

#### 3.0. Main Content

#### 3.1. Legendry Function

##### 3.1.1 Legendry Polynomial

#### 4.0. Conclusion

#### 5.0. Summary

#### 6.0 Tutor Marked Assignment

#### 7.0. References/Further Readings

1.0 Introduction: We shall consider another class of special functions which have wide application in physical problems. These class of functions have Orthogonality properties. The functions are legendry functions.

2.0 Objectives: At the end this unit you should able to  
- identify legendry functions and legendry polynomial

- solve problems relating to legendry functions
- learn about the properties of legendry functions and legendry polynomia

### 3.0 MAIN CONTENT

#### 3.1. Legendry Functions

The Legends differential equation of order n is given by

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$$

The solution of this equation is known as Legendry function

$$(1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + p(p-1)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} \{(n+2)(n+1)c_{n+2} - c_n[n(n+1) - p(p+1)]\}x^n = 0$$

Recurrence relation

$$(n+2)(n+1)c_{n+2} = c_n(n^2 + n - p^2 - p) \quad (2)$$

Thus

$$c_{n+2} = \frac{(p-n)(p+n+1)}{(n+2)(n+1)} c_n$$

Therefore

$$c_2 = \frac{p(p+1)}{21} c_0 \quad c_3 = \frac{p(p+1)}{31} c_0$$

$$c_4 = \frac{p(p-2)(P+3)}{21} c_2 = \frac{(p-2)(p)(p+1)(P+3)}{4.1} c_0$$

$$c_5 - \frac{(p-3)(p+4)}{5.4} c_2$$

$$\frac{(p-3)(p-1)(p+2)(p+4)}{4.1} c_0 \text{ etc}$$

$$y_1 = 1 - p(p+1) \frac{x^2}{21} + (p-2)p(p+1)(p+3) \frac{x^4}{41} \text{-----}$$

$$y_2 = x - (p-1)(p+2) \frac{x^3}{31} + (p-3)(p-1)(p+2)(p+4) \frac{x^5}{51} \text{-----}$$

$$P_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2n-2k)!}{2^n k! n-k! n-2k!}$$

Where  $m$  is the largest integer and run greater the  $\frac{n}{2}$ .

In particular

$$p_0(x) = 1 \quad p_1(x) = \frac{1}{2}(3x^2 - 1)$$

$$p_4(x) = \frac{1}{8}(35x^2 - 30x^2 - 3)$$

$$p_1(x) = x$$

$$\int_{-1}^1 p_m(z) p_n(z) dz = 0 \text{ if } m \neq n$$

Solution:

$$(1-z^2) p_n p_m'' - p_m p_n'' - qz \{ p_n p_m' - p_m p_n' \}$$

$$= [b(n+1) - m+1] P_m P_n = 0$$

and subtracting, we have

$$(1-z^2) \frac{d}{dz} p_n p_m'' - p_m p_n'' - 2z \{ p_n p_m' - p_m p_n' \}$$

$$= [n(n+1) - m(m+1) p_n p_n]$$

$$(1-z^2) \frac{d}{dz} p_n p_m'' - p_m p_n'' - 2z \{ p_n p_m' - p_m p_n' \}$$

Integrate from  $-1$  to  $1$  we have

$$[n(n+1) - m(m+1)] \int_0^1 p_m p_n dz$$

$$(1-z^2) (p_n p_m' - p_m p_n') \Big|_{-1}^1 = 0$$

$$\text{ii } \int_{-1}^1 p_m(z) p_n(z) dz = \frac{2}{2n+n}, \text{ if } m = n$$

Solution:

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} p_n(z)t^n$$

Square is

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} p_n(z)t^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_m(z) p_n(z) t^{m+n}$$

Integrating from -1 to 1

$$\int_{-1}^1 \frac{dz}{1-2zt+t^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \int_{-1}^1 p_m(z) p_n(z) dz \right\} t^{m+n}$$

$$\sum_{n=0}^{\infty} \left\{ \int_{-1}^1 p_n(z) [p_n(z)]^2 dz \right\} t^{2n}$$

$$\text{L.H.S} = -\frac{1}{2t} \log(1-2zt+t^2) \Big|_{-1}^1 = \frac{1}{t} \ln \frac{1+t}{1-t}$$

$$\sum_{n=0}^{\infty} \left\{ \frac{2}{2n+1} \right\} t^{2n}$$

Equating the Coefficients, we have

$$\text{(iii)} \quad (n+1)p_{n+1}(2) - (2n+1)zp_n + np_{n-1}(z) = 0$$

Solution: Differentiating with respect to t both sides of the identity

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} p_n(z)t^n$$

Multiply by  $1-2z+t^2$ , we have

$$(z-t) \sum_{n=0}^{\infty} p_n(z)t^n = (1-2z+t^2) \sum_{n=0}^{\infty} p_n(z)t^n$$

Or

$$\sum_{n=0}^{\infty} zp_n(z)t^n - \sum_{n=0}^{\infty} p_n(z)t^{n+1} = \sum_{n=0}^{\infty} p_n(z)t^n - \sum_{n=0}^{\infty} 2zp_n(z)t^n + \sum_{n=0}^{\infty} p_n(z)t^{n+1}$$

Equating the Coefficient problem: Show that

$$(1 - 2xz + z^2)^{-y^2} = p_0(x) + p_1(x)z + p_2(x)z^2 + \dots \sum_{n=0}^{\infty} p_n(x)z^n$$

Proof:-

$$(1 - 2xz + z^2)^{\frac{1}{2}} = 1 + \frac{1}{2}(2xz - z^2) + \frac{1}{2} \frac{3}{2} (2xz - z^2)^2 + \dots + \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) \dots [(2p-1)2]}{2^p p!} (2xz - z^2)^p$$

The power of  $z^p$  can only occur in the term going from the  $p$ th term  $(2xz - z^2)^p$  [=  $z^p(2x - z)$  down. Thus expanding the various powers of  $(2x - z)$ , we find that the Coefficient of  $z^p$  is

$$\frac{(\frac{1}{2})(\frac{3}{2}) \dots [(2p-1)2]}{2^p p!} (2x)^p$$

Prove that

$$p_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

$$p_n(z) = \sum_{r=0}^n \frac{(-1)^r (2n-2r)}{2^n r!(n-r)!(n-2r)!} z^{n-2r}$$

Where  $p$  is  $\frac{1}{2}n$  or  $\frac{1}{2}(n-1)$ .

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} \sum_{r=0}^{\infty} \frac{(-1)^r n!}{r!(n-r)!} z^{2n-2r}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} \sum_{r=0}^{\infty} \frac{(-1)^r n!}{r!(n-r)!} z^{2n-2r}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

### 3.1.1 Legendry Polynomial

The equation  $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

is called Legendry equation

(i)  $x = +1$  are the regular singular points of the equation. We solve the equation what the singular point  $x=1$ , we put  $x=1=u$  and obtain the transformed equation

(ii)  $u(u+2)\frac{d^2y}{du^2} + n - (n+1)y = 0$  is the regular singular point.

We assume the solution point

$$y = \sum_{k=0}^{\infty} a_k u^{k+c}$$

$$\frac{dy}{du} = \sum_{k=0}^{\infty} a_k u^{k+c} u^{k+c-1}$$

$$\frac{d^2y}{du^2} = \sum_{k=0}^{\infty} (k+c)(k+c-1)a_k u^{k+c} u^{k+c-2}$$

The roots of the indicial equations are  $c = 0, 0$ . Hence one solution is logarithmic. We are only interested here in the non- logarithmic solution. Hence

$$y = \sum_{k=0}^{\infty} a_k u^k$$

We assume  $a_0$  is non-zero arbitrary constant, and

$$a_k \frac{-(k-n-1)(k+n)}{2k^2} a_{k-1}$$

Solving the recurrence solution, we have

$$a_k = \frac{(-1)^k (-n)_k (1+n)_k a_0}{2^k (k!)^2}$$

Thus the solution is

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (-n)_k (n+1)_k (x-1)^k}{2^k (k!)^2}$$

Where  $a_0 = 1$

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)_k (n+1)_k}{(1)_k (k!)^2} \left(\frac{1-k}{2}\right)^k$$

$$y_1 = {}_2F_1(-n, n+1; 1; \left(\frac{1-x}{2}\right))$$

$$= P_n(x).$$

$P_n(x)$  is called the Legendry Polynomials

It is customary to take

$$c_n = \frac{2n!}{2^n (n!)^2}, n = 0, 1, 2, \dots$$

But from (3)

$$c_{n-2} = -\frac{n(n-1)}{(2)(2n-1)}, \quad c_{n-2}, \text{ or}$$

$$c_n = -\frac{(n)(2n-1)}{n(n-1)}, \quad c_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)}$$

$$c_{n-4} = \frac{(2n-4)}{2n2!(n-2)(n-4)}$$

or

$$c_{n-2k} = \frac{(2n-2k)!(-1)^k}{2nk!(n-k)(n-2k)!}$$

Then the legendry Polynomials of degree n is given by

$$P_n(x) = \sum_{k=0}^M \frac{(-1)^k (2n-2k)!}{2^n k!n-k!n-k!n-2k!} x^{n-2k}$$

integer not greater than  $\frac{n}{2}$ .

$$(1) P_n(x) = \sum_{k=0}^M \frac{(-1)^k}{2^n k!(n-k)!} \frac{d}{dx^{n-1}} (x^{2n-2k-1})$$

Since

$$\begin{aligned} \frac{d}{dx^n} (x^{2n-2k}) &= (2n-2k) \frac{d}{dx^{n-1}} (x^{2n-2k-1}) \\ &= (2n-2k)(2n-2k-1)\dots(n-2k+1)x^{n-2k} \\ &= \frac{2n-2k!}{n-2k!} x^{n-2k} \end{aligned}$$

Hence

$$P_n(x) = \sum_{k=0}^M \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^M \frac{(-1)^k n! (x^2)^{n-k}}{k!(n-k)!}$$

We may now extend the range of this sum by letting  $k$  range from 0 to  $n$ . This extension will not affect the result, since the added terms are a polynomial of degree less than  $n$  and the  $n$ th derivative will vanish.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^n \frac{(-1)^k n! (x^2)^{n-k}}{k!(n-k)!}$$

and by binomial theorems, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 0, 1, 2, \dots$$

this is known as Rodrigues formula

Example Show that

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Solution: By Rodrigues' formula

$$\begin{aligned} P_2(x) &= \frac{1}{2!} \frac{d^2}{dx^2} (x^2 - 1) = \frac{1}{2} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) \\ &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

Problem: Show that

$$(i) \quad P_n' + 1(x) = (2n+1)P_n(x) + P_{n-1}'(x), \quad n = 1, 2, \dots \quad (1)$$

$$(ii) \quad P_n' + 1(x) = xP_n'(x) + (n+1)P_n(x). \quad (2)$$

Solution

$$\begin{aligned} (i) \quad P_{n+1}'(x) &= \left[ \frac{d}{dx} \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^{n+1} \right] \\ &= \frac{d}{dx} \left[ \frac{1}{2^n n!} \frac{d^n}{dx^n} [x(x^2 - 1)^n] \right] \quad (n+1)(n^2 - 1)^n 2x \\ &= \frac{1}{2^{n+1}} = \frac{d^{n+1}}{dx^{n+1}} [x(x^2 - 1)^n] \\ &= \frac{1}{2^{n+1}} = \frac{d^n}{dx^n} [x(x^2 - 1)^n] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{m+1}} = \frac{d^n}{dx^n} [(2n+1)(x^2-1)^n + 2n(x^2-1)^{n-1}] \\
&= [(2n+1)P_n(x) - 1] + P'_{n-1}(x)
\end{aligned}$$

**Solution (ii)**

Now we have that

$$\begin{aligned}
&\frac{d}{dx} [f(x)(x) + f'(x)] \\
&\frac{d^2}{dx^2} [xf(x)] = f''(x) + 2f'(x)
\end{aligned}$$

and in general

$$\frac{d^{p+1}}{dx^{p+1}} [xf(x)] = (x) \frac{d^{p+1}}{dx^{p+1}} [x(x^2-1)^n]$$

Now

$$\begin{aligned}
&p'_{n+1}(x) \frac{1}{2} + \frac{d^{n+1}}{dx^{p+1}} \frac{d^{n+1}}{dx^{n+1}} [x(x^2-1)^n] \\
&\frac{1}{2^n n!} [x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (n+1) \frac{d^n}{dx^n} + (n+1) \frac{d^n}{dx^n} (x^2-1)^n] \\
&= xp'_{n+1} + (n+1)p_n(x)
\end{aligned}$$

Eliminating  $p'_{n+1}$ , we have  $np(x) = p'_n - p'(x) - p'_{n-1}, n = 1, 2, \dots, \dots (3)$

**Finally**

$$\begin{aligned}
&(n+1)p_{n+1}(x) - (2_{n+1})xp_n(x) + p_{n-1}(x) \\
&= \frac{xp'_{n+1}(x) - p'_n(x)}{(3)} - \frac{x(p'_{n+1}(x) - p'_{n+1}(x))}{(1)}
\end{aligned}$$

**Thus**

$$-(n+1)p_{n+1}(x) + np_{n-1}(x) = (2n+1)xp_n(x)$$

$$py'' + \theta y' + Ry = 0 \tag{A}$$

$$p\mu + (2p' - \theta)\mu' + (p'' - \theta' + R)\mu = 0 \tag{B}$$

$$p\mu'' + (2p' - 2p' - \theta)\mu' + (p'' - 2p'' - \theta') + p'' - \theta + R)\mu = 0$$

4.0. Conclusion: You have learnt about legendry polynomial and legendry functions in this unit. Read this unit properly before going to the next unit

5.0 Summary: You will recall that the legendry polynomial is defined as

$$P_n(x) = \sum_{k=0}^M \frac{(-1)^k (2n-2k)!}{2^n k! n-k! n-k! n-2k!} x^{n-2k}$$

This polynomial has Orthogonality property which we have mentioned in this unit. Read this unit properly before proceeding to the next unit.

## 6.0 Tutor Marked Assignment

(1) Show that the substitution  $t = 1 - x$  transform Legendre's equation to the form:

$$t(2-t)\frac{d^2y}{dt^2} + 2(1-t)\frac{dy}{dt} + p(p+1)y = 0$$

(2) Problem: Show that

(i)  $P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x)$ .  $n = 1, 2, \dots$

(ii)  $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$ .

## 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

## UNIT2: Some Examples of Partial Differential Equations

1.0 Introduction

2.0 Objectives

3.0 Main Content

3.1. Some Examples of Partial Differential Equations

4.0 Conclusion

5.0 Summary

6.0 Tutor Marked Assignment

7.0 References/Further Readings

1.0 Introduction: A partial differential equation is an equation that contains one or more partial derivatives. Such equations occur frequently in application of mathematics. We shall only discuss certain partial differential equations which are used frequently in applied mathematics. In fact we are going to discuss a kind of boundary value problems which enters modern applied mathematics at every turn.

2.0 At the end of this unit you should be able to

- recognize partial differential equations by type and character
- learn about methods of solving partial differential equations
- apply the knowledge in some other related field.

### 3.0 MAIN CONTENT

#### 3.1. Some Examples of Partial Differential Equations in Applied Mathematics:

Many linear problems in applied mathematics involve the solution of an equation obtained by specializing the form.

$$\Delta^2 \theta + f = \lambda \frac{d^2 \theta}{dt^2} + \mu \frac{d\theta}{dt} \quad (1)$$

Where  $f$  is a specified function of position and  $\lambda$  and  $\mu$  are certain specified physical constant. Here  $\Delta^2$  is the Laplacian operator in one, two or dimension under consideration and is of the form.

$$\Delta^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \quad (2)$$

In rectangular co-coordinator of three space. The unknown function  $\phi$  is the function of the position co-ordinates  $(x, y, z)$  and the time  $t$ .

#### (i) Laplace equation

$$\Delta^2 \theta = 0 \quad (3)$$

It is satisfied, by the velocity potential in an ideal incompressible fluid without vertical or continuously distributed sources and by gravitational potential in free space, electrostatic potential in the steady flow of electric currents in solid conductors, and by the steady-state temperature distribution in solids.

#### (ii) Poisson's equation

$$\Delta^2 \theta + f = 0 \quad (4)$$

is satisfied, for example by the velocity potential of an incompressible. Irrotational, ideal fluid with continuously distributed sources or by steady temperature, distribution due to distributed heat sources, and by a “sheds function” involved in the elastic torsion of prismatic bars, with a suitably prescribed function  $f$ .

#### (iii) Wave equation

$$\Delta^2 \theta = \frac{1}{c} \frac{\partial^2 \theta}{\partial t^2} \quad (5)$$

arises in the study of propagation of waves with velocity  $c$ , independent of the wave length. In particular, it is satisfied by the components of the electric or magnetic vector in electromagnetic theory, by suitably chosen component of displacement, in the theory of elastic vibrations, and by the velocity potential in the theory of sound (acoustics) for a perfect gas.

**(iv) The equation of heat conduction**

$$\Delta^2 \theta = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t} \quad (6)$$

is satisfied, for example, by the temperature at a point of a homogeneous body and by the concentration of a diffused substance in the theory of diffusion, with a suitably prescribed constant  $\theta$ .

**(v) The telegraphic equation**

$$\frac{\partial^2 \theta}{\partial x^2} + \lambda \frac{\partial \theta}{\partial t} + \mu \frac{\partial \theta}{\partial t} \quad (7)$$

Which is one dimensional specialization of (1), is satisfied by the potential in a telegraph cable, where  $\lambda = Lc$  and  $\mu = Rc$ , if the Leakage is neglected ( $L$  is inductance,  $c$  capacity and  $R$  resistance Per unit length).

(vi) Differential equation of higher order, involving the operator  $\Delta^2$ ,, also are rather frequently encountered, in particular, the bi-Laplacian equation in two dimensions.

$$\Delta^4 \theta = \Delta^2 \Delta^2 \theta = \frac{\partial^4 \theta}{\partial x^4} + 2 \frac{\partial^2 \theta}{\partial x^2 \partial y^2} + \frac{\partial^4 \theta}{\partial y^4} = 0 \quad (8)$$

is involved in many two dimensional problem of the theory of Elasticity.

The solution of a given problem must satisfy the proper differential equation, together with similarly prescribed boundary condition or initial conditions (2 f time is involved).

The above equation can be changed to cylindrical co-ordinates  $r, \theta, z$ , related to  $x, y$  and  $z$  by the equations

$$x = r \cos \theta, y = r \sin \theta, z = z$$

$$\Delta^2\theta = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{2} \frac{\partial\theta}{\partial r} + \frac{1}{2} \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2\theta}{\partial\theta^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \quad (9)$$

In spherical co-ordinates  $P, \theta, \theta$  related to  $x, y, z$  by the equations

$x = P\sin\theta\cos\theta, y = P\sin\theta, z = P\cos\theta$  Laplace' equation is

$$\frac{\partial^2 v}{\partial p^2} + \frac{2}{p} \frac{\partial v}{\partial p} + \frac{\partial^2 v}{\partial\theta^2} + \frac{\cot\theta}{p^2} \frac{\partial v}{\partial\theta} + \frac{\cos^2\theta}{p^2} \frac{\partial^2 v}{\partial\theta^2} = 0. \quad (10)$$

In what now follows we shall solution methods of partial Differential Equations:

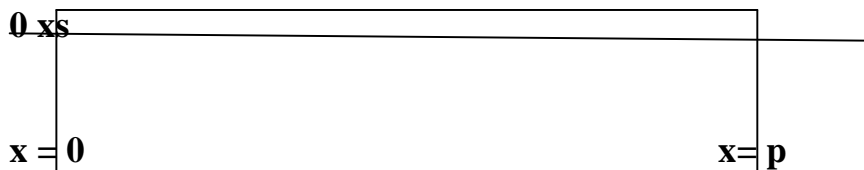
Method of Separation of variables

Consider the equation

$$\alpha^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < l, t > 0 \quad (a)$$

This is called the heat conduction equation

**Straight bar**



This is a straight bar of uniform cross section and homogenous material. The temperature  $v$  can be considered constant on any given cross section.

$v = U(x, t)$ .

$\alpha^2$  is a constant known as  $v = U(x, t)$  is the thermal diffusivity. In addition, we shall assume that the ends  $v = U(x, t)$  of the bar are held at temperature zero: thus  $v = 0$  when and  $x = l$ .

$$\therefore u(0, t) = 0, \quad u(l, t) = 0, \quad t > 0, \quad (1)$$

Finally, the initial distribution of temperature in the bar is assumed to be given, thus  $U(x, 0) = h(x) \quad 0 \leq x \leq l \quad (2)$

(1) and (2) are called boundary conditions

We assume that

$$u(x, t) = f(x)g(t) \quad (3)$$

Substituting equation (3) for  $u(x, t)$  in (1) yields

$$\alpha^2 f''(x)g(t) = f(x)g'(t) \quad (4)$$

or

$$\alpha^2 \frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)} \quad (5)$$

Now equation (5) is said to have its variable separated; that is, the left member of equation (5) is a function of  $x$  alone and the right member of equation (5) is a function of  $t$  alone.

Since  $x$  and  $t$  are independent variables, the only way in which a function of  $x$  alone can equal a function of  $t$  alone is for each function to be constant.

$$\therefore \frac{f''(x)}{f(x)} = b^{\theta} \quad (6)$$

$$\alpha^2 \frac{g'(t)}{g(t)} = b \quad (7)$$

In which  $b$  is arbitrary

The partial differential equation (1) has now been replaced by two ordinary differential equations. This is the essence of the **method of separation of variables**.

**Boundary Conditions:**

$$xv(0,t) = f(0)g(t) = 0 \quad (8)$$

by (1), if  $g(t) = 0$  then  $u_{x,t}$  will be identically zero, it is not acceptable because it does not satisfy the equation (2). Thus it must satisfy the condition

$$f(0) = 0 \quad (9)$$

Similarly, the boundary condition at  $x(l)$   $U(l,t) = 0$  requires

$$f(l) = 0 \quad (10)$$

There are two possible values of the constant  $k$  i.e.  $k = 0$  or  $k \neq 0$ .

Values of the constant  $k$  :

(i)  $k = 0$ , then the general solution of equation (6) is

$$f(x) = c_1 + c_2 \quad (11)$$

(11) Must satisfy the boundary value conditions (9) and (10). In order to satisfy (9)

$$f(0) = c_1 + c_2 \implies c_2 = 0 \quad (12)$$

It is also satisfies the equation (10)

$$\therefore f(l) = c_1 l = c_1 = 0 \text{ Since } l \neq 0.$$

$$\therefore c_1 = 0 \quad (13)$$

Hence  $f(x)$  is identically zero, and therefore  $U(x,t)$  is also identical zero

(ii)  $k \neq 0$ , we take  $k = -\lambda^2$ , where  $\lambda$  is a new parameter. Thus the equation (6) becomes

$$f''(x) + \lambda^2 f(x) = 0 \quad (14)$$

and its general solution is

$$f(x) = b_1 e^{i\lambda x} + b_2 e^{-i\lambda x} \quad (15)$$

Applying the boundary condition (9) and (10), we have

$$b_1 + b_2 = 0 \quad (16)$$

$$\left. \begin{aligned} f(x) = b_1 e^{i\lambda x} + b_2 e^{-i\lambda x} = 0 \end{aligned} \right\}$$

The system (16) has a non-trivial solution  $k_1 = 0$  and  $k_2 = 0$  always, but it is not acceptable  $u(x,t)$  is identically zero. Non-trivial solution exists if and only if the determinant

$$\begin{vmatrix} i e^{i\lambda l} & i e^{-i\lambda l} \\ 1 & 1 \end{vmatrix} = 0 \quad (17)$$

If we write,  $\lambda = \mu + i\nu$  then

$$\begin{aligned} e^{-\nu l} e^{i\mu l} - e^{i\mu l} e^{-\nu l} &= 0 \quad \text{or} \\ e^{-\nu l} (\cos \mu l - i \sin \mu l) - e^{-\nu l} (\cos \mu l + i \sin \mu l) &= 0 \\ \left. \begin{aligned} \cos \mu l (e^{-\nu l} - e^{-\nu l}) &= 0 \\ \sin \mu l (e^{-\nu l} + e^{-\nu l}) &= 0 \end{aligned} \right\} \quad (18) \end{aligned}$$

Now  $\cos \mu l (e^{-\nu l} - e^{-\nu l}) > 0$  for  $\nu$  and  $l$ , thus  $\sin \mu l = 0 \Rightarrow \mu = 0$  (19) must be so chosen that

$$\mu = \frac{n\pi}{l}, \quad (20)$$

where  $n$  is a non-zero integer. From (16)  $k_1 = -k_2$  (21)

From (15), we have

$$f(x) = b_1 (e^{-i\mu x} e^{\nu x} - e^{i\mu x} e^{-\nu x}) = \frac{b_1}{2} (e^{-i\mu x} e^{\nu x} - e^{i\mu x} e^{-\nu x}) \quad (23)$$

$$k = -\lambda^2 = -\frac{n^2 \pi^2}{l^2} \quad (24)$$

Where  $n$  is an integer

$$\text{From (7), we have} \quad (25)$$

Hence the function

$$u_n(x,t) = c_n \exp\left[-\frac{n^2 \pi^2 \alpha^2 t}{l^2}\right] \sin \frac{n\pi x}{l} \quad (26)$$

$n = 1, 2, 3, \dots$  where  $c_n$  is an arbitrary constant, satisfy the boundary conditions 2,9,10 as well as the differential equation (1). The functions  $u_n$  are sometimes called fundamental solution of the Heat Conduction problem (a) (1) and (2).

By the boundary condition (2) we get from (26).

$$u_n(x,0) = c_n \sin \frac{n\pi x}{l} \quad (27)$$

For  $n = 1, 2, \dots$

Each solution given by (27) satisfy the differential equation and the boundary condition. Since partial differentials equation involved is linear and homogeneous in  $u$  and its derivatives, a sum of solution are also a solution. From the known solutions,  $u_1, u_2, \dots, u_n, \dots$ . We may thus construct other with sufficiently strong convergence condition it is true that even the infinite series.

$$u = \sum_{n=1}^{\infty} u_n \text{ or}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2 \pi^2 \alpha^2 t}{l^2}\right) \sin \frac{n\pi x}{l} \quad (28)$$

is a solution of the differential equation. In order to satisfy the initial condition (2) we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} h(x) \quad (29)$$

Now let us suppose that it is possible to express  $h(x)$  by means of an infinite series forms

$$h(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad (30)$$

We know how to compute  $b_n$  i.e

We can satisfy the equation (29) by choosing  $c_n = b_n$ . For each  $n$ . With the coefficient selected in this manner, equation (28) gives the solution of the boundary value problem (a) (1) and (2)

Thus we have solved the problem consisting of the heat condition equation.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < l, \quad t > 0 \quad (1)$$

The boundary condition

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0 \tag{2}$$

and the initial condition

$$u(x,0) = h(x), \quad 0 \leq x \leq l \tag{3}$$

we found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left[-\frac{n^2\pi^2\alpha^2 t}{l^2}\right] \sin\frac{n\pi x}{l} \tag{4}$$

With the coefficients  $b_n$  are the same as in the series

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{5}$$

Where

$$b_n = \frac{2}{l} \int_0^l h(x) \sin\frac{n\pi x}{l} dx \tag{6}$$

The series in equation (5) is just the Fourier

Example 2. Now we consider the problem of the heat conduction equation of boundary conditions and the initial condition the boundary conditions are known as non-homogeneous boundary condition.

Solution: we shall reduce the present problem to one having homogeneous boundary condition we use the physical argument. After a long time, i.e. ...., we anticipate a steady state temperature distribution ..... will be reached, and must satisfy difficulties (1), then (which is independent of time t and initial condition).

..... (4)

and it satisfied the boundary condition

..... (5)

Which apply even as ..... The solution (4) with condition (5)

..... (6)

Hence the steady state temperature is a linear function of  $x$ .

We shall express  $U(x,t)$  as the sum of the steady state temperature and another distribution  $w(x, t)$ .

$$\therefore U(x,t) = U(x) + w(x,t) \quad (7)$$

(7) satisfies (1), we have

$$\alpha^2 (u + w)_{xx} = (u + w)_t \quad (8)$$

It follows that

$$\alpha^2 W_{xx} = W_t \quad (9)$$

Now boundary condition

$$w(0,t) = u(0,t) - u(0) = T_1 - T_1 = 0 \quad (10)$$

$$w(l,t) = u(l,t) - u(l) = T_2 - T_2 = 0 \quad (11)$$

The initial condition

$$w(x,0) = (u(x,0) - u(x)) = f(x) - u(x) \quad (12)$$

Where  $u(x)$  is given by (6)

The problem now becomes precisely the previous one and we have the solution

$$W(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2 \pi^2 \alpha^2 t}{l}\right] \sin \frac{n\pi x}{l} \quad (13)$$

Where

$$b_n = \frac{2}{l} \int_0^l W(x,0) \sin \frac{n\pi x}{l} dx \quad (14)$$

Where

$$U(x,t) = (T_2 - T_1) \frac{x}{l} + T_1 + \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2 \pi^2 \alpha^2 t}{l}\right] \sin \frac{n\pi x}{l} \quad (15)$$

Where

$$b_n = \frac{2}{l} \int_0^l \left[ f(x) - (T_2 - T_1) \frac{x}{l} - T_1 \right] \sin \frac{n\pi x}{l} dx \quad (16)$$

**Example 3** Now we consider the problem of the heat conduction equation with the boundary condition

$$\alpha^2 U_{xx} = U_x \quad (1)$$

$$V_x(0,t) = 0, \quad V_x(l,t) = 0, \quad t > 0 \quad (2)$$

and the initial condition

$$v(x,0) = f(x) \quad (3)$$

**Solution:** We solve the equation by the method separation of variables. [when the ends of the bar are insulated so that there is no passage of heat through them].

$$V(x,t) = h(x)g(t) \quad (4)$$

(4) satisfies (1), we have

$$\frac{h'(x)}{h(x)} = \frac{1}{\partial^2} \frac{g'(t)}{g(t)} - \alpha \quad (5)$$

We assume that  $\alpha$  is real, we consider three cases  $\alpha = 0$  and  $-ve$ .

(i) If  $\alpha = 0$  then equation (5) given

$$V(x,t) = K_1x + K_2 \quad (6)$$

Applying boundary condition (2), we get

$K_1 = 0$ . Hence corresponding

$$h'(x) = 0$$

$$h(x) = C_1 + C_2$$

$$g'(t) = 0 \rightarrow g(t) = C_B \quad (7)$$

Solution is

(ii) If  $\alpha = \lambda^2$  where  $\lambda$  is real and  $+ve$

$$\therefore h''(x) - \lambda^2 h(x) = 0 \quad (8)$$

$$g'(t) - \lambda^2 h(x) = 0 \quad (9)$$

From (8) and (9), we have

$$U(x,t) = e^{\alpha^2 \lambda^2 t} (k_1 \text{Sin} \lambda x + k_2 \text{Cosh} \lambda x). \quad (10)$$

Now we apply the boundary condition (2)

$$U_x(x,t) = e^{\alpha^2 \lambda^2 t} (k_1 \lambda \text{Cos} \lambda x + K_2 \text{Sinh} \lambda x)$$

$$U_x(0,t) = 0 \quad \text{and} \quad U_x(l,t) = 0$$

$$K_2 \lambda \text{Sin} \lambda l = 0 \Rightarrow K_2 = 0$$

$$K_2 \lambda \sin \lambda l = 0 \Rightarrow K_2 = 0$$

$$\rightarrow K_1 = 0 \text{ and } K_2 = 0$$

This is not acceptable, because it does not satisfy the initial condition. Some Examples of Partial Differential Equation hence  $\sigma$ , can not be positive.

(iii)  $\sigma = -\lambda^2$ , where  $\lambda$  is +ve and real. From (8) and (9), we obtain

$$U_x(x,t) = e^{\alpha^2 \lambda^2 t} (k_1 \lambda \cos \lambda x + K_2 \cos \lambda x) \quad (11)$$

Now we apply boundary condition, we get (x=0)  $K_1 = 0$  and (x=e)  $\lambda \frac{\lambda}{e}$  for  $n=1,2,\dots$   $\sin \theta = 0, \theta = n\pi$

$$\therefore \sigma = -\left(\frac{n^2 \pi^2}{e^2}\right), \text{ where } n \text{ is +ve integer} \quad (12)$$

Combining the solution, we have

$$U_o(x,t) = \frac{1}{2} C_o \quad (13)$$

$$U_n(x,t) = C_n \exp\left[-\frac{n^2 \pi^2 \alpha^2 t}{e^2}\right] \cos \frac{n\pi x}{e}, n = 1,2,\dots \quad (14)$$

These solution functions satisfy the differential equation (1) and bounding conditions (2) for any value of the constant  $C_n$ . both differential equation and boundary values are linear and homogeneous, any finite sum of the fundamental solutions will also satisfy them. We will assume that this is also line for convergent infinite sums of fundamental solution also as well.

Thus

$$U(x,t) = \frac{1}{2} C_o + \sum_{n=1}^{\infty} c_n \exp\left[-\frac{n^2 \pi^2 \alpha^2 t}{e^2}\right] \cos \frac{n\pi x}{e} \quad (15)$$

Where  $C_n$  are determined by the initial requirement that

$$U(x,o) = \frac{1}{2} C_o + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{e} = f(x) \quad (16)$$

Thus the unknown coefficients in equation (15) must be coefficients in the Fourier Cosine series of period  $2e$  for  $f$ . Hence

$$U_n \frac{e^l}{e^o} \int_0^l f(x) \cos \frac{n\pi x}{e} dx, n = 0, 1, 2, \dots$$

With this choice of the coefficients, (15) provides the solution of the equation.

Example: Elastic string with non-zero initial displacement

First suppose that string is displaced from its equilibrium position, and then released with zero velocity at time  $t = 0$  to vibrate freely. Then in vertical displacement  $U(x,t)$  must satisfy the wave equation.

$$X^2 U_{xx} = U_{tt} \quad 0 < x < l, t > 0 \quad (1)$$

The boundary conditions are

$$U(0,t) = 0, \quad U(l,t) = 0, \quad t \geq 0 \quad (2)$$

and the initial conditions

$$U(x,0) = f(x), U_t(x,0) = 0, \quad 0 \leq x \leq l \quad (3)$$

Where  $f$  is given function describing the configuration of the string at  $t = 0$

Solution: We use the equation (1) by the method of separation of variables.

Assuming that

$$U(x,t) = X(x) T(t) \quad (3)$$

Substituting  $u$  in (1), we get

$$\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T''}{T} = \sigma \quad (4)$$

We assume that  $\sigma$  is real (we shall prove it some where else). We consider these cases  $\sigma = 0$ , -ve and +ve.

(i)  $\sigma = 0$ , then  $X'' = 0$ , and  $X(x) = K_1 x + K_2$  (5)

(ii) If  $\sigma > 0$ , then  $X'' - \lambda^2 x = 0$  and  $X(x) = K_1 \sinh \lambda x + K_2 \cosh \lambda x$  (6)

Where  $\lambda = \sqrt{\sigma}$

Consider the solution given by (5)

$$U(x,t) = (K_1 x + K_2) T(t)$$

By boundary conditions  $U(0,t) = 0$

$U(0,t) = (0 + K_2)T(t) = 0$ ,  $T(t)$  can not be zero, because  $U(x,t)$  will be identically zero. Thus  $K_2 = 0$ . Next consider the second boundary condition

$U(l,t) = 0$  then  $U(l,t) = (K_1 l)T(t) = 0 \Rightarrow K_1 = 0$

Thus  $X(x) = 0$ , it is not acceptable.

(iii) Similarly, we can show that for (6) under boundary condition  $K_1 = 0 = K_2$ .

Thus  $\sigma = 0$  and  $\sigma = +ve$  real number are not acceptable. We now consider the last case

$$(ii) \quad \sigma = -\lambda x = 0 \quad (7)$$

$$X'' + \lambda X = 0 \quad (8)$$

$$T'' + \lambda^2 \alpha^2 T = 0$$

$$\therefore X(x) = K_1 \sin \lambda x + K_2 \cos \lambda x \quad (9)$$

$$T(t) = K_3 \sin \lambda \alpha t + K_4 \cos \lambda \alpha t \quad (10)$$

Thus

$$U(x,t) = (K_1 \sin \lambda x + K_2 \cos \lambda x)(K_3 \sin \lambda \alpha t + K_4 \cos \lambda \alpha t) \quad (11)$$

Satisfies (1) for all values of  $K_1, K_2, K_3, K_4$  and for  $\lambda > 0$ .

Now we impose the boundary conditions

$U(0,t) = 0$ , Thus

$$U(0,t) = K_2(K_3 \sin \lambda \alpha t + K_4 \cos \lambda \alpha t) \Rightarrow K_2 = 0 \quad (12)$$

Secondly boundary condition  $U(l,t) = 0$ ,

$$U(l,t) = (K_1 \sin \lambda l)(K_3 \sin \lambda \alpha t + K_4 \cos \lambda \alpha t) = 0 \quad (13)$$

If  $K_1 = 0$ , then  $U(x,t)$  is zero identically, thus for non-trivial solution.

$$\sin \lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, \dots \quad (14)$$

Hence the functions which satisfy the equation (1) and boundary condition (2) are of the form.

$$U_n(x,t) = \sin \frac{n\pi x}{l} \left( C_n \sin \frac{n\pi \alpha t}{l} + K_n \cos \frac{n\pi \alpha t}{l} \right) \quad (15)$$

Where  $n=1, 2, \dots$ ,  $C_n$  and  $K_n$  are arbitrary constants. Now we apply the principle of superposition of solution and assume that

$$U_n(x,t) = \sum_{n=1}^{\infty} U_n(r,t)$$

$$\sum_{n=1}^{\infty} \text{Sin} \frac{n\pi x}{l} (C_n \text{Sin} \frac{n\pi \alpha t}{l} + K_n \text{Cos} \frac{n\pi \alpha t}{l}) \quad (16)$$

Further we assume that (16) can be differentiated term by term with respect to t

$U_t(x,0) = 0$  yields

$$U_n(x,0) = \sum_{n=1}^{\infty} C_n \frac{n\pi \alpha}{l} \text{Sin} \frac{n\pi x}{l} = 0 \quad (17)$$

$\Rightarrow C_n = 0$  for all values of n

$\Rightarrow$  The other condition  $U(x,0) = f(x)$

Given

$$U_n(x,0) = \sum_{n=1}^{\infty} K_n \text{Sin} \frac{n\pi a}{l} = f(x) \quad (18)$$

Consequently  $K_n$  must be line coefficients in the Fourier Series of period  $2l$  for f and are given by

$$K_n \frac{2}{l} \int_0^l f(x) \text{Sin} \frac{n\pi a}{l} dx, n = 1, 2, \dots \quad (19)$$

Thus the formal solution of the problem (1) with condition (2) and (3) is

$$U_n(x,t) = \sum_{n=1}^{\infty} K_n \text{Sin} \frac{n\pi a}{l} \text{Cos} \frac{n\pi \alpha x}{l} \quad (20)$$

Where the coefficients  $K_n$  are given by (19).

For a fixed value of n the function

$\text{Sin} \frac{n\pi a}{l} \text{Cos} \frac{n\pi \alpha x}{l}$  is periodic in time t with the period  $\frac{2l}{n\alpha}$ ; it therefore represents a vibratory motion of the string having this period or having the frequency  $\frac{n\pi \alpha x}{l}$ . The quantities  $\lambda a = \frac{n\pi a}{l}$  for  $n=1, 2, \dots$  are the natural

frequencies of the string. The factor  $K_n \text{Sin} \frac{n\pi a}{l}$  represents the displacement pattern occurring in the string, when it is executing vibrations of the given frequency.

In the case of heat conduction problem, it is attempting to try to show this by directly substituting equation (20) for  $U(x, t)$  in (1), (2) and (3), we compute.

$$U_{xx}(x, t) = -\sum_{n=1}^{\infty} K_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad (21)$$

Due to the presence of the factor  $n^2$  in the numerator, this series may not converge. It is not possible to justify by direct with respect to either variable in  $0, l$  and  $t > 0$ , provided  $h$  is twice continuously differentials on  $(-\omega, \omega)$ . This require  $f'$  and  $f''$  are continuous on  $[0, l]$ . Furthermore, since  $h''$ , we must have

$$f''(0) = f''(l) = 0$$

Example: General Problem for inelastic string.

Consider the equation

$$\alpha^2 U_{xx} - U_{tt} = 0, \quad 0 < x < l, \quad t > 0; \quad (1)$$

The boundary condition

$$U(0, t) = 0, \quad U(l, t) = 0 \quad (2)$$

and the initial conditions

$$U(x, 0) = f(x), U_t(x, 0) = g(x), 0 \leq x \leq l \quad (3)$$

Solution: As we have done in the previous case. We obtain the solution

$$U_{xx}(x, t) = \sum_{n=1}^{\infty} U_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} C_n \sin \frac{n\pi \alpha t}{l} + K_n \cos \frac{n\pi \alpha t}{l} \quad (4)$$

Applying the initial condition  $U(x, 0) = f(x)$  yields

$$U(x, 0) = -\sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{l} = f(x) \quad (5)$$

Where the coefficients  $K_n$  are given in the Fourier Sine Series of period  $2l$  for  $f$  and are given

$$K_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, n = 1, 2, \dots \quad (6)$$

Differentiate (4) with respect to  $t$  and putting substitution. We establish the validity in a different way. We show

$$U(x, t) = \frac{1}{2} [ h(x - \alpha t) + h(x + \alpha t) ] \quad (22)$$

Where  $h$  is function obtained by extending the initial data  $f(x)$  into  $(-l, 0)$  as an odd function, and other values of  $x$  as a periodic function on period  $2l$ .

That is

$$h(x + 2l) = h(x).$$

Now

$$h(x) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{l}$$

Then

$$h(x - \alpha) = \sum_{n=1}^{\infty} K_n \left( \sin \frac{n\pi x}{l} \cos \frac{n\pi \alpha}{l} - \cos \frac{n\pi \alpha}{l} \sin \frac{n\pi \alpha}{l} \right)$$

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} C_n \sin \frac{n\pi \alpha}{l} + K_n \cos \frac{n\pi \alpha}{l}$$

Equation h is the function (20) follows on adding the two equation.

Note: [If  $f(x)$  has a Fourier series, then it must be periodic and continuous].

1. If  $U(x,t)$  is continuous for  $0 < x < l$  and  $t > 0$  provided that h is requires that f is continuous on line interval  $(-\infty, \infty)$ . This requires that f is continuous on the interval  $(0,l)$  and, since h is odd periodic extension of f, that f be zero at  $x=0$  and  $x=p$ .
2. U is twice continuously differentiable  $t=0$ , we get

$$U_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{e} C_n \sin \frac{n\pi a}{l} = g(x) \quad (7)$$

Hence the coefficient  $\vartheta \left( \frac{n\pi a}{l} \right) C_n$  are the coefficients in the Fourier Sine Series of period  $2l$  for  $g$ ; Thus

$$\frac{n\pi a}{l} = C_n \frac{2}{l} \int_0^l [ g(x) \sin \frac{n\pi x}{l} dx, \quad n=1,2,\dots \quad (8)$$

Thus the equation (4) with the equation (6) and (8) constitutes the formal solution of the equation (1).

Example: Laplace's equation: One of the most important of all the partial differential equation occurring in Applied Mathematics is associated with the name of Laplace. Laplace equation in two dimensions.

$$U_{xx} + U_{yy} = 0 \quad (1)$$

and in dimension

$$U_{xx} + U_{yy} = 0 \quad (2)$$

Now solve (1) under the boundary condition. The problem of finding a solution of Laplace's equation which takes on given boundary values is known as Dirichlet problem.

Problem I. Solve the Laplace equation

$$U_{xx} + U_{yy} = 0 \quad (1)$$

In the rectangle  $0 < x < a$ ,  $0 < y < b$ , and which satisfies the boundary condition

$$\begin{aligned} U(x,0) = 0, & \quad U(x,b) = 0, & \quad 0 < x < a, & \quad (2) \\ U(0,y) = 0, & \quad U(a,y) = f(y), & \quad 0 \leq y \leq b \end{aligned}$$

Where  $f$  is given function on  $0 \leq y \leq b$

**Solution:**

$$U(x, y) = X(x)Y(y) \quad (3)$$

Substituting  $U(x, y)$  in (1), we get

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \quad (4)$$

We assume that  $k$  is real.

$$(i) \text{ If } K=0 \text{ then } X'' = 0 \text{ and } Y'' = 0, \text{ and } U(x, y) = (K_1x + K_2)(C_1y + C_2) \quad (5)$$

The homogeneous boundary conditions  $y=0$  and  $y=b$  can be satisfied  $y = C_1 = C_2 = 0 \Rightarrow U(x, y)$ , is identically zero. Hence  $K=0$  is acceptable.

(ii)  $K = \lambda^2, \lambda > 0$ , then

$$X'' = \lambda^2 X = 0$$

$$Y'' = \lambda^2 Y = 0$$

and thus

$$U(x, y) = (K_1 \sinh \lambda x + K_2 \cosh \lambda x)(C_1 \sin \lambda y + C_2 \cos \lambda y) \quad (6)$$

In order to satisfy the boundary conditions  $x = 0$  and  $y = 0 \Rightarrow K_2 = 0 = C_2$

The condition at  $y=b$  becomes

$$K_1 C_1 \sin \lambda x \sin \lambda b = 0 \quad (7)$$

$$\Rightarrow \sin \lambda b = 0$$

It follows that

$$\lambda b = n\pi, \quad n=1,2,3,\dots \quad (8)$$

Thus the solution of the differential equation must be of the forms.

$$U_n(x, y) = C_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \quad n=1,2,3,\dots \quad (9)$$

These functions are the fundamental solutions of the present problems. We assume

$$U_{xx}(x, y) = \sum_{n=1}^{\infty} U_n(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} C_n \sinh \frac{n\pi \alpha x}{b} + \sin \frac{n\pi \alpha x}{b} \quad (10)$$

Now the last boundary conditions

$$U_{xx}(a, y) = f(y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \quad (11)$$

Thus the coefficients  $C_n \sinh \frac{n\pi a}{b}$  must be the coefficients in the Fourier Sine Series of period  $2b$  for  $f(y)$  and are given by

$$C_n \sinh \frac{n\pi a}{b} \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} dy \quad (12)$$

Thus (10) is the solution of the equation (1) satisfying the boundary condition (2) and coefficients  $C_n \sinh \frac{n\pi a}{b}$  are computed from (12).

(iv) If  $K = -\lambda^2$  then

$$X'' + \lambda^2 X = 0$$

$$X'' + \lambda^2 Y = 0$$

and

$$U(x, y) = (K_1 \sinh \lambda x + K_2 \cos \lambda x)$$

$$C_1(x, y) \sinh \lambda y + C_2 \cosh \lambda y \quad (13)$$

Again the boundary condition at  $y=0$  and  $y=b$  lead to  $C_1 = C_2 = 0$ , so again  $U(x, y)$  is zero, everywhere. Hence  $K = \lambda^2$  is not acceptable.

**Problem:** Dirichlet problem for a circle

Consider the Laplace's equation in polar co-ordinates

$$U_{zz} + \frac{1}{z} U_z \frac{1}{2} + U_z \frac{1}{z^2} U_{\theta\theta} \quad (1)$$

With boundary condition

$$U(a, \theta) = f(\theta) \quad (2)$$

$f$  is a given function on  $0 \leq \theta \leq 2\pi$ .

Moreover, in order that  $U(z, o)$  be single valued, it is necessary that, as a function of  $\theta$ ,  $U$  must be periodic with period  $2\pi$ .

$$\text{Solution: Let } U(z, o) = R(z)\theta(\theta) \quad (3)$$

We substitute (3) in (1)

$$r''\theta + \frac{1}{2}R'\theta + \frac{1}{r^2}R\theta'' = 0$$

Or

$$r^2 \frac{R''}{R} + z \frac{R'}{R} = -\frac{\theta''}{\theta} = K \quad (4)$$

Again we assume that the separation constant must be real

(1) Suppose  $K=0$

$$r^2 R'' + rR'\theta = 0$$

$$\theta'' = 0$$

$$\therefore U(r, \theta) = (K_1 + K_2 \text{Log } 2)$$

$$\theta(\theta) = C_1 + C_2\theta$$

$$\therefore U(r, \theta) = (K_1 + K_2 \text{Log } 2)(C_1 + C_2) \quad (5)$$

Since equation is periodic in  $\theta$ , thus  $C_2 = 0$ .

Further  $r \rightarrow 0$  the term  $\log r$  is unbounded this behaviour is unacceptable. Thus we impose the condition that  $U(z, o)$  remains finite at all points of the circle and hence we must take

$$K_2 = 0$$

$$\therefore U_0(r, o) = \text{Constant} = \frac{1}{2}c_0 \text{ say} \quad (6)$$

(ii) If  $K = \lambda^2$  then

$$\theta'' - \lambda^2\theta = 0 \quad (7)$$

$$\therefore \theta(\theta) = C_1 e^{\lambda\theta} + C_2 e^{-\lambda\theta} \quad (8)$$

The function  $U(r, o)$  is periodic thus  $C_1 + C_2 = 0$ .

This makes  $U(r, 0)$  identically zero. This is not acceptable.

(iii) Finally  $K = \lambda^2, \lambda > 0$ , yields

$$r^2 R'' + rR' - \lambda^2 R = 0 \tag{9}$$

And

$$\theta'' + \lambda^2 \theta = 0 \tag{10}$$

$$\therefore R(r) = K_1 r^\lambda + K_2 r^{-\lambda}$$

$$\theta(\theta) = C_1 \sin \lambda \theta + C_2 \cos \lambda \theta \tag{11}$$

In order that  $\theta$  be periodic with period  $2\pi$  it is necessary that  $\lambda$  be a positive integer.

Moreover the solution  $r^{-\lambda}$  of (a) be discarded, since it becomes unbounded as  $r \rightarrow 0$ . Consequently  $K_2 = 0$ . Hence the solutions (1) are

$$U_n(r, \theta) = r^n (C_n \cos n\theta + K_n \sin n\theta), n=1, 2, \dots$$

These functions, together with that of equations (6), serve as fundamental solutions of the present problem. Thus

$$U(r, \theta) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + K_n \sin n\theta) \tag{13}$$

The boundary condition (2) then requires that

$$f(\theta) = U(a, \theta) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} a^n (C_n \cos n\theta + K_n \sin n\theta) \tag{14}$$

for  $0 \leq \theta \leq 2\pi$ .

The function  $f(\theta)$  may be extended outside this interval so also it is periodic of period  $2\pi$ , and has a Fourier Series of the function (14)

$$a^n C_n = \frac{1}{2} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \tag{15}$$

$$a^n K_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \tag{16}$$

With this choice of coefficients (13) represents the solutions of the boundary value problem of equations (1) and (2).

(1) The Heat conduction equation in two space dimension may be expressed in terms of polar co-ordinates as

$$\alpha^2 (U_{zz} + \frac{1}{2} U_{\theta\theta}) = U_t$$

Assuming that

$$U(r, \theta, t) = R(r)\theta(\theta)T(t). \text{ Find ordinary equation satisfied by } R(r), \theta(\theta)T(t)$$

The integrand is an even function.

$$= \int_0^c (1 - \cos \frac{2n\pi x}{c}) dx$$

$$= [ x - \frac{C}{2n\pi} \sin \frac{2n\pi x}{c} ]_0^c = C. \quad n=1,2,\dots$$

Similarly we can obtain

$$I_5 = \int_{-c}^c \cos^2 \frac{n\pi x}{c} dx = C \quad \text{for } n=1,2,3,\dots$$

$$= 2C \quad \text{for } n=0.$$

Similarly we can show that

$$I_4 = \int_{-c}^c \sin^2 \frac{n\pi x}{c} \sin \frac{k\pi x}{c} dx = C \quad \text{if } k = n$$

$$I_5 = \int_{-c}^c \cos \frac{n\pi x}{c} \cos \frac{k\pi x}{c} dx = C \quad \text{if } k = n$$

Periodic function: A function  $f$  is said to be periodic with period  $T$  if the domain of  $f$  contains  $x+T$  whenever it contains  $x$ , and  $y$ .

$$f(x+T) = f(x) \quad \text{for every value of } x.$$

$$f(x+T) = f(x)$$

With fundamental period  $T = 2l/m$ . Every such function has the period (2l)

The function  $\sin^2 \frac{m\pi x}{l}$  and  $\cos \frac{m\pi x}{l}$ ,  $n = 1,2,\dots$  are periodic.

### 3. Fourier Series

We assume that there exists a series expansion of the type

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [ a_n \cos \frac{n\pi x}{C} + b_n \sin \frac{n\pi x}{C} ] \quad (1)$$

Valid in the interval  $-C \leq x \leq C$

$$j_p(z) = (\frac{z}{2})^b \sum_{m=1}^{\infty} \frac{(-1)^m}{m!m+p!} (\frac{z}{2})^{2m}$$

(1) is called the Fourier Series corresponding to  $f(x)$ ,  $a_n$  and  $b_n$ .

Multiply (1) by  $\sin(\frac{k\pi x}{C}) dx$ , where  $k$  is a true integer, and then

integrate each term from  $-c$  to  $c$ , thus arriving at

$$\int_{-c}^c f(x) \sin \frac{k\pi x}{C} dx = \frac{1}{2} a_0 \int_{-c}^c \frac{n\pi x}{C} dx + \sum_{n=1}^{\infty} [ a_n \int_{-c}^c \cos \frac{n\pi x}{C} \sin \frac{k\pi x}{C} dx ] \quad (2)$$

As seen earlier

$$\int_{-c}^c \cos \frac{n\pi x}{C} \sin \frac{k\pi x}{C} dx = 0 \text{ for all } k \text{ and } n$$

And

$$\int_{-c}^c \sin \frac{n\pi x}{C} \sin \frac{k\pi x}{C} dx = 0 \text{ if } k \neq n$$

$$=c \text{ if } k = n$$

Using in (2), we have

$$\int_{-c}^c f(x) \sin \frac{b\pi x}{C} dx = \frac{cbk}{e}, K= 1,2,3, \dots$$

or

$$b_n \frac{1}{c} a_0 \int_{-c}^c f(x) \sin \frac{n\pi x}{C} dx, n=1,2, \dots$$

Let us now evaluate the coefficients  $a_n$ . Using the multiplies

$\cos \frac{k\pi x}{C} dx$  throughout equation (1) and then integrating term by term for  $-c$  to

$c$ , we get

$$\int_{-c}^c f(x) \cos \frac{k\pi x}{C} dx = \frac{1}{2} a_0 \int_{-c}^c \cos \frac{k\pi x}{C} dx + \sum_{n=1}^{\infty} [ a_n \int_{-c}^c \cos \frac{n\pi x}{C} \cos \frac{k\pi x}{C} dx + b_n \int_{-c}^c \sin \frac{n\pi x}{C} \cos \frac{k\pi x}{C} dx ]$$

(4)

Now we know that

$$\int_{-c}^c \cos \frac{n\pi x}{C} \cos \frac{k\pi x}{C} dx = 0 \text{ for } n \neq k$$

$$=c \text{ for } n = k$$

If  $k \neq 0$ , (4) reduces to

$$\int_{-c}^c f(x) \cos \frac{k\pi x}{C} dx = ea_k$$

or

$$a_n \int_{-c}^c \cos f(x) \cos \frac{n\pi x}{C} dx \tag{5}$$

Next we determine the coefficient  $a_n$ . Suppose  $K=0$  in (4)

$$\int_{-c}^c f(x) dx = \frac{1}{2} a_0 \int_{-c}^c f(x) dx + \sum_{n=1}^{\infty} [ [ a_n \int_{-c}^c \cos \frac{n\pi x}{C} dx + b_n \int_{-c}^c \sin \frac{n\pi x}{C} dx ] ] \tag{6}$$

Thus we have

$$\int_{-c}^c f(x)dx = \frac{1}{2}a_0(2c)$$

or  $\int_{-c}^c a_0 = \frac{1}{c} \int_{-c}^c f(x)dx$  (7)

Thus we write the formal expansion as follows

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{C} + b_n \sin \frac{n\pi x}{C} \right] \quad (8)$$

With

$$a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{C} dx, n = 0, 1, \dots \quad (9)$$

$$b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{C} dx, n = 1, 2, \dots \quad (10)$$

Note that the formulae (9) and (10) depend only upon the values of f(x) in the interval  $-c \leq x \leq c$ . Since each of the terms in the Fourier series (8) is periodic with period 2c, the series converges for all x whenever it converges in  $-c \leq x \leq c$ , and its sum is also a periodic function with period 2c. Hence f(x) is determined for all x by its values in the interval  $-c \leq x \leq c$ .

4.0 Conclusion: You have been introduced to partial differential equation in this unit. The attempts here are just introductory. You required to study this unit properly because you will need them in your subsequent courses in mathematics.

5.0. Summary: In this units various forms and types of partial differential equations were studied: These include (1) Wave equation (2) Laplace Equation (3) Heat Equation. e.t.c.

We also proposed various methods of solving these equations which include method of separation of variables and Fourier series applications. You are required to study this unit properly and attempt all the exercises at the end of the unit.

6.0 Tutor Marked Assignment:

(1) Show that the boundary-value problem

$$\frac{d^2 y}{dx^2} - k^2 y = 0, y(0) = y(l) = 0$$

Cannot have a nontrivial solution for real values of k

(2) Determine those values of  $k$  for which the partial differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Possesses nontrivial solutions of the form  $T(x, y) = f(x) \sinh ky$  which vanish when  $x = 0$ , and, when,  $x = l$

(3) By considering the characteristic functions of the problem

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0,$$

Show that  $\int_{-1}^1 P_r(x) P_s(x) dx = 0$

#### 7.0. REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

